OPTIMAL DESIGN OF MULTI-PARAMETER DYNAMIC SYSTEMS

by

Erkal Tüzgiray

United States
Naval Postgraduate School

THESIS

OPTIMAL DESIGN OF MULTI-PARAMETER
DYNAMIC SYSTEMS

by

Erkal Tüzgiray

October 1969

Optimal Design of Multi-Parameter Dynamic Systems

by

Erkal Tüzgiray
Lieutenant, Turkish Navy
B.S., Naval Postgraduate School, 1968

Submitted in partial fulfillment of the
requirements for the degree of

ELECTRICAL ENGINEER

from the

NAVAL POSTGRADUATE SCHOOL
October 1969

ABSTRACT

Two design methods for multi-parameter dynamic systems are proposed. They are intended to eliminate the limitations and disadvantages of the existing design methods. The powerful mathematical tools of optimal control theory are applied to the practical design problems of classical control.

The first method is intended for linear systems only; the design problem is solved in the s-domain, by finding "the best root locations" of the system's characteristic equation. In the second method, the design problem is solved by finding "the best response" of the system in the time domain. The second method is applicable to a wide range of dynamic systems; it can be used to synthesize linear, nonlinear and sampled-data systems, and systems with time delay. This method is also extended to a numerical stability analysis procedure.

Fourteen examples are presented to illustrate the applications of the methods.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENT

The author acknowledges with gratitude the advice and guidance of Professors George J. Thaler and Donald E. Kirk.

I. INTRODUCTION

Classical control theory and modern control theory have been extended in diverging routes since the inception of modern control theory. Modern techniques have been successful for larger scale problems and for problems which include some new performance criteria, such as expenditure of control energy, etc., but a general purpose or a special purpose computer in the control loop is usually necessary to implement the solution.

Most design techniques of classical control theory were developed before the extensive application of digital computers to system design problems. Using a digital computer in the design phase of a control system is usually feasible, especially if the design can be implemented by using passive circuit elements rather than complicated and expensive memory units.

During the last decade mathematical methods and numerical techniques have been developed and applied to the problems of "Optimal Control Theory". The same methods have found application in other areas of science, but they have not been applied efficiently to the design procedures of the dynamic systems which have been defined in the sense of classical control theory. The classical techniques, although having many drawbacks and limitations (see Section II.A), were well developed by the time effective optimization methods arrived on the scene. The classical methods have been successful for a restricted class of design problems; for this reason probably, modern control theory grew independently, rather than improving and replacing the existing methods.

In this thesis it is intended to narrow the gap between the classical and modern control theories. Powerful mathematical tools of optimization theory are applied to the practical design problems of classical control. Two general design methods for multi-parameter dynamic systems are proposed:

1. Optimization for the Best Root Location in the s-Domain, and

2. Optimization for the Best Response in the Time Domain.

Chapter II of the thesis is devoted to the first method which is intended for linear systems only -- the design problem is solved in the s-domain by finding "the best root locations" of the system's characteristic equation. Four linear system design examples are included. The rest of the thesis is devoted to the second method and its various applications. In Chapter III, a general description and the philosophy of the method "Optimization for the Best Response in the Time-Domain" is given, and it is applied to the same linear system design problems discussed in Chapter II, to provide a basis for comparison of the results obtained by both methods.

The time-domain method is applicable to a large variety of system design problems. To demonstrate this point, Chapter IV is devoted to various applications of the method. Three examples for nonlinear and sampled-data systems and a system with transport lag are presented in Chapter IV.

In Chapter V, a numerical method to investigate the stability limit of the dynamic systems is introduced as an extension of the time-domain method.

In the year 1964, the gap between control theory and the practical control problems became substantial and extensive research effort has been directed to bridge this gap between the theory and paractice (see, for example, Ref. 2). The methods presented in this thesis can be thought of as another approach to suboptimal system design problems. In this respect, these methods can be considered as applications and extensions of Optimal Control Theory; on the other hand, these methods provide better solutions to the practical dynamic system synthesis problems of Classical Control and they can be considered as the extensions of Classical Control. Since the extensions from Optimal Control Theory and Classical Control are toward each other, these methods hopefully provide one of the necessary bridges to reduce the gap.

II. LINEAR SYSTEM DESIGN BY OPTIMIZATION FOR THE BEST ROOT LOCATIONS IN THE S-DOMAIN

A. GENERAL

Present design methods of classical control theory are limited and have some serious disadvantages. Root-locus and frequency-domain techniques, for example, can handle systems which have only one parameter. Algebraic methods (in the coefficient and parameter planes) are two-variable procedures. The major design tools of these methods are a set(s) of graphs, and obtaining these graphs for higher-order systems is quite a laborious task. The trial-and-error nature of the classical design procedures is, however, the most significant limitation of these methods.

Since the dynamic behavior of second-order systems has been thoroughly studied and is well known, most of the present methods for high-order system design depend on the dominant-roots concepts. Finding a pair of complex dominant roots for a high-order system is equivalent to approximating the system by a second-order system.

A pair of complex roots of high-order system can be made dominant by satisfying at least one of the two conditions given below:

1. Transients in the time response due to undesired roots (all the roots except the selected "dominant" pair) last for a much shorter time than the transients of the selected pair. This is achieved if the time constants of the

undesired roots are much smaller than those of the selected pair. In the s-domain, the magnitudes of the real parts of the undesired roots must be much larger than the magnitudes of the real parts of the selected pair to satisfy this condition. The remote undesired roots may cause undesirable effects in the time response for a short period of time (near zero time), but thereafter the time response is governed by the selected roots.

2. The coefficients in the time response associated with the desired roots are much larger than those associated with the undesired roots. In other words, the contribution of the selected pair to the transient response is much larger than the contribution of the other roots. In the s-domain, this condition is satisfied if the residues of the desired roots are much larger than the residues of the other roots.

If both of these conditions are satisfied, the time response of the selected pair dominates the overall time response, and the selected pair of roots is called "dominant". If only one of the conditions is satisfied an acceptable transient response may still be obtained. The first condition alone, for example, has been used in the s-domain design procedures to avoid the lengthy and time-consuming residue calculations at each step of the trial-and-error procedures.

When designing a high-order feedback system by root-locus methods, or by algebraic methods, the designer usually selects a pair of complex root locations in the s-domain, and then tries to locate two of the roots of the system's characteristic equation at these selected locations and the other (undesired) roots at remote points of the left-hand side of the s-plane. If this can be achieved, the transient response of the system closely follows the selected (desired) second-order system's response. There are usually other specifications to be met, such as the steady-state error, maximum allowable gain, and limitations on the system components, which are treated as additional constraints.

The number of free independent parameters (N_c) cannot exceed the order of the characteristic equation (n), and if the number of free independent parameters is equal to the order of the system ($N_c = n$), the roots of the characteristic equation can be located at any point in the s-plane. The design is completed by simple algebraic manipulations (see, for example, Ref. 12). If the number of free independent parameters is less than the order of the characteristic equation of the system ($N_c < n$), however, the designer loses his control over some of the roots. Since one free parameter exactly locates one root, when all parameters are used to locate an equal number of roots, the excess ($n - N_c$) root(s) may be anywhere in the s-plane. Some of these roots may move very close to the origin and become dominant, or some may even go into the right half of the s-plane and make the system unstable.

Two parameters can be used to locate a pair of complex roots exactly at the specified locations, but these roots can be constrained to move on a line by using just one parameter. For example, their real parts (σ) or imaginary parts (ω) can be fixed, or they can be forced to move on a constant damping ratio (ζ)-line or on a constant natural frequency (ω_n)-circle.

When designing with a fewer number of parameters than the degree of the characteristic equation, the procedure is a trial-and-error method and success may depend primarily on the past experience of the designer. There are several methods for locating two of the roots of the characteristic equation at the desired locations, but if the number of free parameters is less than the order of the system, none of these methods ensure

a. the stability of the system, and

b. the dominance of the selected roots.

B. PHILOSOPHY OF OPTIMIZATION

Since placing a pair of complex roots at the desired locations has no significance without ensuring their dominance and the system's stability, the method proposed in the first part of the thesis provides these two conditions by confining the undesired roots to appropriate region(s) of the s-plane, and then finds the optimum parameter values to locate the dominant roots as close to the specified locations as possible[1]. Since the undesired roots are not placed at specific locations (they are only confined to stay in a remote region of the s-plane) none of the free parameters are "used up"; only their freedom is limited.

The problem is solved in the "Parameter Space" which is a N_c-dimensional Euclidean space. A performance index (J), which is usually a measure of closeness of the dominant roots to the desired root locations, is defined. By minimizing the performance index with respect to the selected free parameters ($\underset{\sim}{z}$) (see Section II.D) one obtains the "best possible solution" to the design problem as it is specified. The design specifications yield some linear and nonlinear constraining equations and/or inequalities; it is assumed that these constraining equations define a bounded region in the N_c-dimensional parameter space. Some of the constraining equations are generally nonlinear and it may sometimes be impossible to find a feasible point (a point which simultaneously satisfies all of the constraining equations and inequalities) to minimize the performance index. In practical terms, this means that the design (or compensation) is not always attainable by using a certain

---

[1] For the simplicity of explanation it is assumed here that the design is carried out for a selected pair of complex roots; but this is not mandatory. The method is generalized (in Section II.B) and an example is given (Example 3) for the design of systems without a pair of selected dominant roots.

type of compensator. If this occurs, the designer should change the system configuration (usually by changing the type of compensator, or by inserting additional compensation).

C. CONSTRAINTS

Performance specifications and available system components dictate the constraints, which can be divided into three general groups:

1. Constraints due to physical limitations, such as limitations on the available system components.

2. Constraints due to static (or steady-state) performance specifications.

3. Constraints due to dynamic performance specifications.

The constraints in the first two groups are linear or can be approximated by linear inequalities. The constraints in the last group are obtained by factoring the system's characteristic equation (as explained in the next section) and they are expressed as "n" linear and/or nonlinear simultaneous algebraic equations.

D. FACTORING THE CHARACTERISTIC EQUATION

The denominator of the system's transmission function (from input to output) is called "the characteristic polynomial", and when it is equated to zero it is called "the characteristic equation". The characteristic polynomial of the system is a n^{th}-degree polynomial in the complex variable "s", and a function of N_c free system parameters ($\underset{\sim}{\alpha}$) ; that is:

$$P_{ch}(s) = \sum_{i=0}^{n} a_i \, s^i \qquad , \tag{II.1}$$

where $a_n \triangleq 1$ and other coefficients are either known constants or functions of one or more system parameters. Therefore, the characteristic polynomial can be represented as

$$P_{ch}(s) = P_{ch}(\underset{\sim}{\alpha}, s) \; . \tag{II.2}$$

To represent the desired and undesired roots explicitly, it is necessary to factor the characteristic polynomial into two parts. In this factored form the characteristic polynomial may be written as

$$P_{ch}(\underset{\sim}{\alpha}, s) = \left[P_d(\sigma, \omega, s) \right] \cdot \left[P_{ud}(\underset{\sim}{\beta}, s) \right] \tag{II.3}$$

where $\underset{\sim}{\beta}$ represents the coefficients in P_{ud} (a polynomial of degree "n-2" in s, representing the undesirable roots) and P_d is a quadratic in s which represents the desired roots. Two forms of P_d will be considered:

$$P_d(\sigma, \omega, s) = s^2 + 2\sigma s + (\sigma^2 + \omega^2) , \tag{II.4}$$

where σ and ω are the real and imaginary parts of the desired complex roots, and

$$P_d(\zeta,\omega_n,s) = s^2 + 2\zeta\omega_n s + \omega_n^2, \tag{II.5}$$

where ζ is the damping ratio and ω_n is the natural frequency of the second-order system.

The characteristic equation can be factored by using several methods, such as, by dividing the characteristic equation by P_d or P_{ud} and equating the remainder (in each case) to zero. In another method the general forms of P_d and P_{ud} are multiplied together and the coefficients of the product polynomial are equated to the corresponding coefficients of the characteristic polynomial; that is, the identity shown below is solved

$$P_d \cdot P_{ud} \equiv P_{ch} . \tag{II.6}$$

These processes yield n algebraic equations, all of which must be satisfied to obtain the desired factored form of the characteristic equation.

The left side of Eq. (II.3) contains N_c free system parameters ($\underset{\sim}{\alpha}$), and the right side is a function of n root parameters ($\sigma,\omega\ \underset{\sim}{\beta}$); that is, there are a total of N_c + n parameters available, but only N_c of them are independent--the others are related to the independent ones through the n algebraic constraining equations which are obtained during the factoring process. Any N_c of these parameters can be chosen as free variables ($\underset{\sim}{z}$); the remaining n variables are determined by the constraining equations due to dynamic performance specifications (see Paragraph II.C(3) above).

E. THE PERFORMANCE INDEX (J)

When the characteristic equation is factored, that is, after the free variables ($\underset{\sim}{z}$) are chosen and the constraining equations are found, a performance index (J) is defined so that when it is minimized with respect to the chosen variables ($\underset{\sim}{z}$) in the bounded region defined by the constraining equations and inequalities, it yields "the best root locations". The best root locations may mean that the desired roots are as close to the specified locations as possible (if the dominant roots concept is being used in the design), or to satisfy other criteria -- as explained below -- and that the undesired roots are in remote regions of the left-hand side of the s-plane (see Figs. II.1 and II.2).

The form of the performance index depends on the design specifications. If the design is to be carried out by locating a pair of dominant roots as close to the desired locations as possible, then the performance index may take the form

$$J = (\sigma_d - \sigma)^2 + (\omega_d - \omega)^2 \quad , \quad \text{or} \tag{II.7}$$

$$J = (\zeta_d - \zeta)^2 + (\omega_{n_d} - \omega_n)^2 \tag{II.8}$$

where the subscript "d" indicates the desired coordinates.

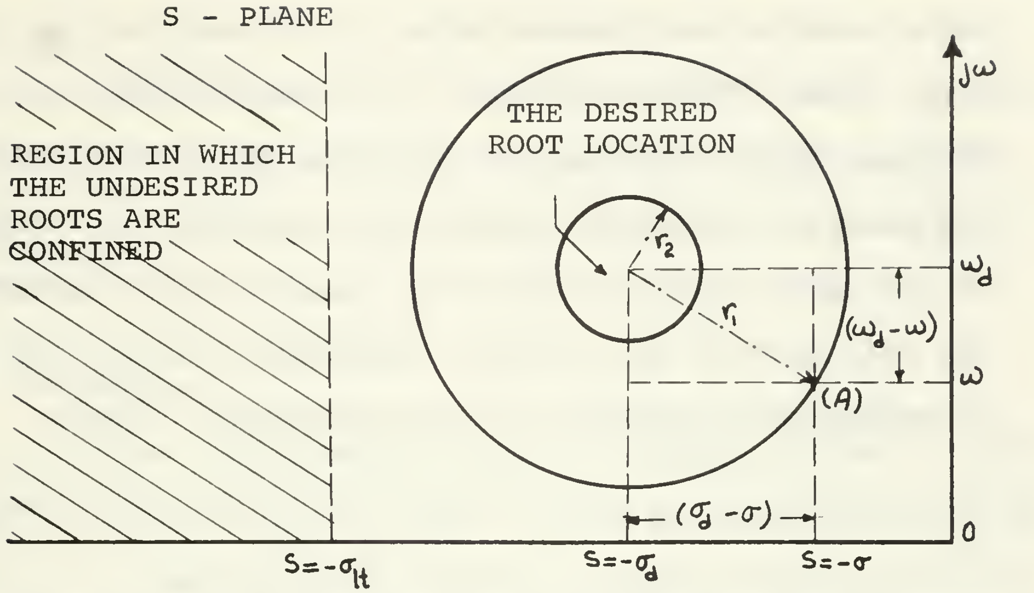


Fig. II.1. The desired and undesired root location in the s-plane.

NOTES: Point (A) is an actual root location at a minimization step ($J = r_1^2$)

r_2 is a predetermined norm and it is equivalent to ϵ in Fig. II.5.

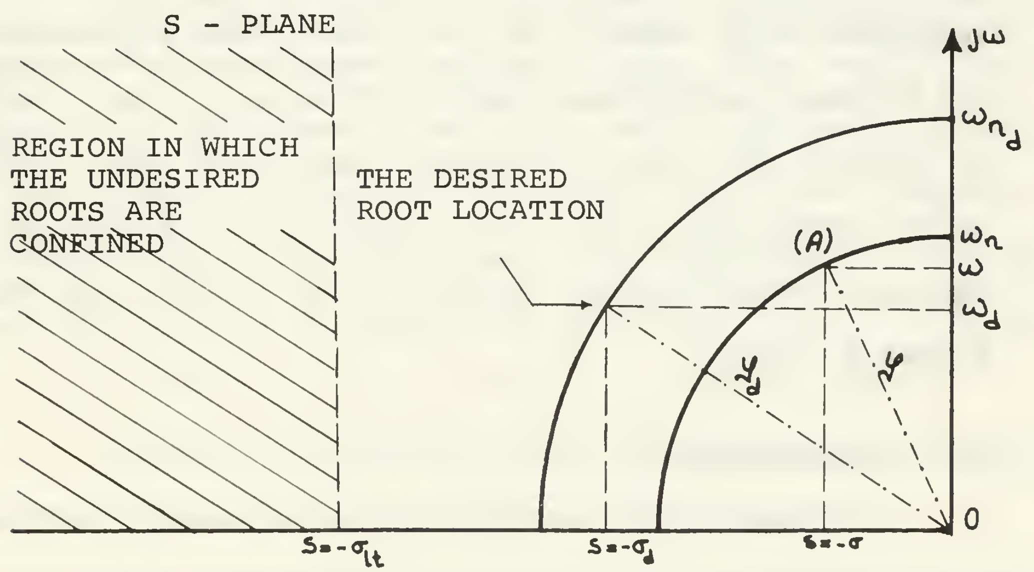


Fig. II.2 The desired and undesired root locations in the s-plane.

(Point (A) is an actual root location at a minimization step).

Sometimes design specifications allow the designer to use a simpler performance index. For example, only the first terms of Eq. (II.7) or (II.8) may be used. For some purposes extra terms may be added to the performance index. For example, it may be desired to confine the desired roots to two small regions around the desired root locations and the real part of the closest undesired root (to the origin) is maximized by minimizing the performance index.

$$J = -q \quad , \quad (q > 0) \tag{II.9}$$

where q is the magnitude of the real part of the closest undesired root. The negative sign in (II.9) converts the maximization problem into a minimization problem.

For different kinds of dynamic performance specifications, different performance indices can be defined and minimized with respect to the chosen free parameters. If the dynamic performance specifications are very tight (that is certain rise time, overshoot, settling time, frequency of the transients, etc., are desired) it is best to use the dominant-roots concept (Eq.(II.7) or (II.8)) for the performance index.

F. PROGRAMMING

Although it is possible to solve simple problems by hand calculations, in general, a computer program capable of minimizing a nonlinear function with linear and nonlinear constraints is necessary for higher-order systems. For the examples considered in this part of the thesis SUBROUTINE BOXPLX (see Refs. 1 and 4) has been used in the minimization with good results. A computer program which is written in FORTRAN IV, and used for the minimization part of Example - 4 is given at the end of the thesis (see COMPUTER PROGRAM I).

G. EXAMPLE 1

A linear third-order feedback control system is shown in block diagram form in Fig. II.3(a). A step-by-step procedure to compensate the system for the desired transient response by using the method "Optimization for the Best Root Locations in the s-Domain" is given below.

1. <u>Interpret the Desired Dynamic Performance Specifications</u>

The desired dynamic performance specifications are usually given as maximum overshoot, rise time, settling time, frequency of the transients, etc., and it is usually best to find a pair of complex root locations in the s-domain to represent these specifications. For this problem it is assumed that a pair of complex roots located at $\sigma_d = 0.5$ and $\omega_d = \pm\ 0.5$ would yield the desired dynamic response.

2. <u>Check to Determine if the Desired Roots are on the Root Locus of the Uncompensated System</u>

The designer gains considerable insight into the design problem by just sketching the root locus; this also helps in deciding on the type of compensation that should be used, if different choices are available. The root locus for the uncompensated system is shown in Fig. II.3(b), and the desired locations are not on the root locus.

3. Decide on the Type of Compensation

A cascade compensator will be used. The system with the cascade compensator is shown in Fig. II.4(a).

4. Find the Transmission Function and the Characteristic Equation

The compensated system becomes fourth order (n=4), and there are three free system parameters (N_c = 3) -- the pole location (p), the compensation ratio (k) of the compensator, and the system gain (K). The vector representing the free system parameters is

$$\underset{\sim}{\alpha} = (k\ p\ K)^T \ . \tag{II.10}$$

The transmission function of the compensated system is

$$T(s) = \frac{C(s)}{R(s)} = \frac{(K/k)(s + kp)}{s(s+1)^2(s+p) + (K/k)(s+kp)} \ , \tag{II.11}$$

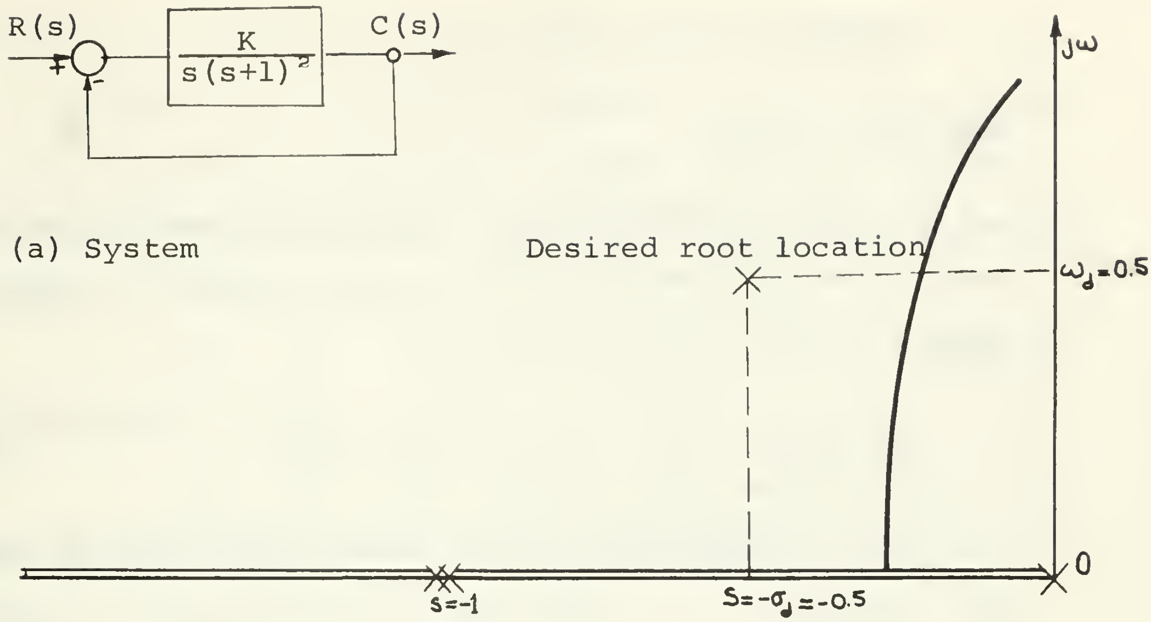
and the characteristic equation is

$$s^4 + (2+p)s^3 + (1+2p)s^2 + (p+K/k)s + Kp = 0 \ . \tag{II.12}$$

5. Factor the Characteristic Equation

Using one of the methods described in Section II.D factor the characteristic equation and find:

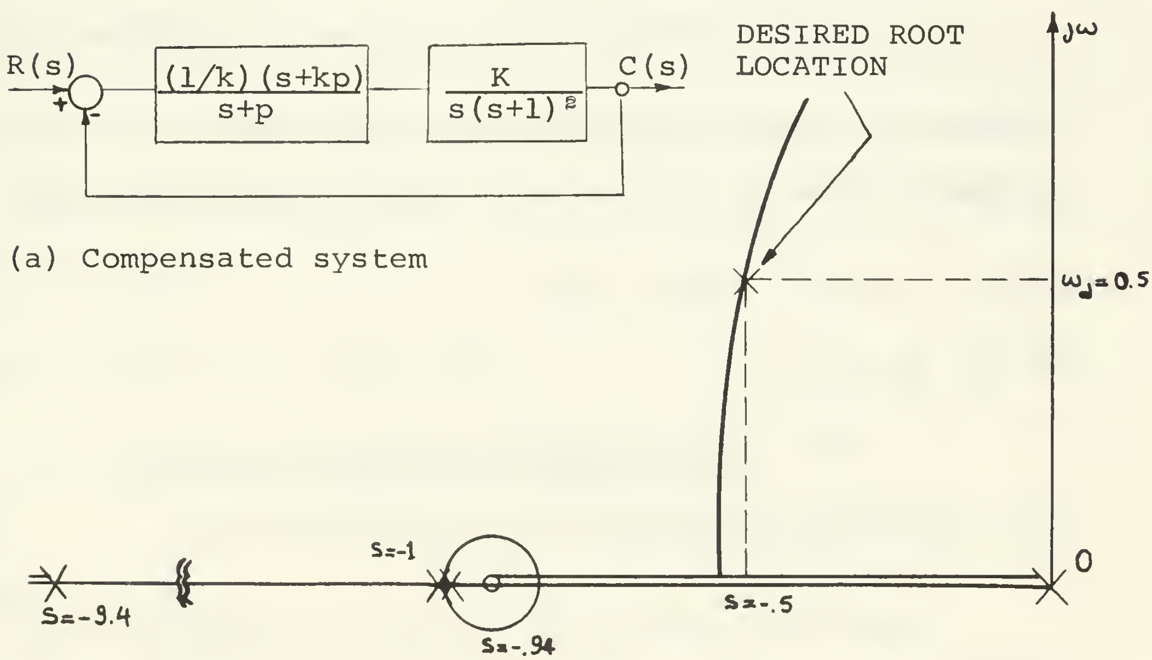
a. The $(n-2)^{th} = 2^{nd}$-degree polynomial (P_{ud}) to represent the undesired roots.



(a) System

(b) Uncompensated system's root locus

Fig. II.3. EXAMPLE 1



(a) Compensated system

(b) Root locus for the compensated system

Fig. II.4. EXAMPLE 1

$$P_{ud}(\underset{\sim}{\beta},s) = s^2 + 2as + b \tag{II.13}$$

where $\underset{\sim}{\beta} = (a\ b)^T$.

b. A quadratic in s (P_d) to represent the desired roots (the roots which will be made dominant by optimization).

$$P_d(\sigma,\omega,s) = s^2 + 2\sigma\ s + (\sigma^2 + \omega^2) \tag{II.14}$$

c. The type -3 constraining equations by using Eq. (II.6)

$$P_{ch}(\underset{\sim}{\alpha},s) = \left[P_d(\sigma,\omega,s)\right] \cdot \left[P_{ud}(\underset{\sim}{\beta},s)\right] . \tag{II.6}$$

For notational convenience let $c \triangleq (\sigma^2 + \omega^2)$,

$$s^4 + (2+p)s^3 + (1+2p)s^2 + (p+K/k)s + Kp \equiv$$
$$s^4 + 2(a+\sigma)s^3 + (c+4a\sigma+b)s^2 + 2(ac+b\sigma)s + bc. \tag{II.15}$$

Equate the coefficients of the polynomials of the above identity and obtain the n(=4) constraining equations

$$p + K/k = 2(ac + b\sigma) \tag{II.16}$$

$$Kp = bc \tag{II.17}$$

$$1 + 2p = 4a\sigma + b + c \tag{II.18}$$

$$2 + p = 2(a + \sigma) \tag{II.19}$$

There are total of $N_c + n$ (= 7) variables (p,K,k,a,b, σ,ω), and any N_c (= 3) of them can be chosen as the free variables to constitute the $\underset{\sim}{z}$-vector of the minimization; the remaining n (= 4) will be related to the free variables by the constraining equations. After selecting the free variables, the constraining equations should be rearranged so that:

(1) the left side of each equation is one of the dependent variables, and

(2) the right side of each equation must contain only the free variables and the dependent variables defined in the preceding equations.

To avoid substitution, σ, a, and c are chosen as the free variables, hence

$$\underset{\sim}{z} = (a \ \sigma \ c)^T , \qquad \text{(II.20)}$$

and the constraining equations are solved for the remaining four variables

$$p = 2\ (a + \sigma - 1) \qquad \text{(II.21)}$$

$$b = 1 + 2p - c - 4a\sigma \qquad \text{(II.22)}$$

$$K = bc/p \qquad \text{(II.23)}$$

$$k = K/(2\ (ac + b\sigma) - p) \qquad \text{(II.24)}$$

6. <u>State the Additional Constraints</u>

a. The compensator will contain only passive elements; hence,

$$p > 0 \qquad \text{(II.25)}$$

b. A physical constraint is assumed on the system gain

$$0.4 \le K \le 0.5 \tag{II.26}$$

c. The compensation ratio of a cascade compensator should not be greater than ten.

$$0.1 \le k \le 10 \tag{II.27}$$

d. To ensure complex dominant roots,

$$\sigma^2 - c < 0 \tag{II.28}$$

e. To confine the undesired roots to the left of $s = -\sigma_{1t} = -1.5$ line,

$$a - (a^2 - b)^{\frac{1}{2}} \ge \sigma_{1t} = 1.5 \tag{II.29}$$

f. All free and dependent parameters must be positive; hence

$$\begin{aligned} \sigma &> 0 \\ c &> 0 \\ b &> 0 \end{aligned} \tag{II.30}$$

7. <u>Define a Performance Index</u>

For this problem the performance index defined by Eq. (II.7) is suitable,

$$J = (\sigma_d - \sigma)^2 + (\omega_d - \omega)^2 \tag{II.7}$$

8. <u>Minimize the Performance Index</u>

Find the best root locations and the corresponding optimum system parameters ($\underset{\sim}{\alpha}^*$) by minimizing the performance index with respect to the chosen free parameters $\underset{\sim}{z} = (a\ \sigma\ c)^T$ in the bounded parameter space defined by the constraining inequalities (II.25) - (II.30) and the constraining Eqs. (II.21) - (II.24).

9. <u>Relax the Necessary Boundaries if Possible</u>

If at the end of a minimization process the selected roots cannot be located in a close neighborhood of the desired locations (determined by a preselected norm -- see Eq. (II.31) below), and the result is a constrained minimum (that is, at least one of the variables is on the limiting boundary) then the limiting boundary(s) should be relaxed (if possible) and the minimization should be repeated. In this example the σ_{1t} value was the only boundary which limited the minimization process. The boundary on the undesired roots was relaxed toward the origin by 0.1 steps until

$$J^{1/2} \leq \epsilon = 10^{-4} \tag{II.31}$$

was obtained. After each relaxation, SUBROUTINE BOXPLX was called and the performance index was minimized again to obtain optimum parameter values for the new boundaries. The results obtained at the end of each minimization are tabulated in Table II.1. Relaxation was stopped when the undesired root (#1) was no longer on the σ_{1t} boundary (see last line of Table II.1); at the same step the stopping criterion on the performance index (Eq. (II.31)) was also satisfied.

The repeated minimization process and boundary relaxation were achieved automatically in the main program (see the computational flow chart -- Fig. II.5).

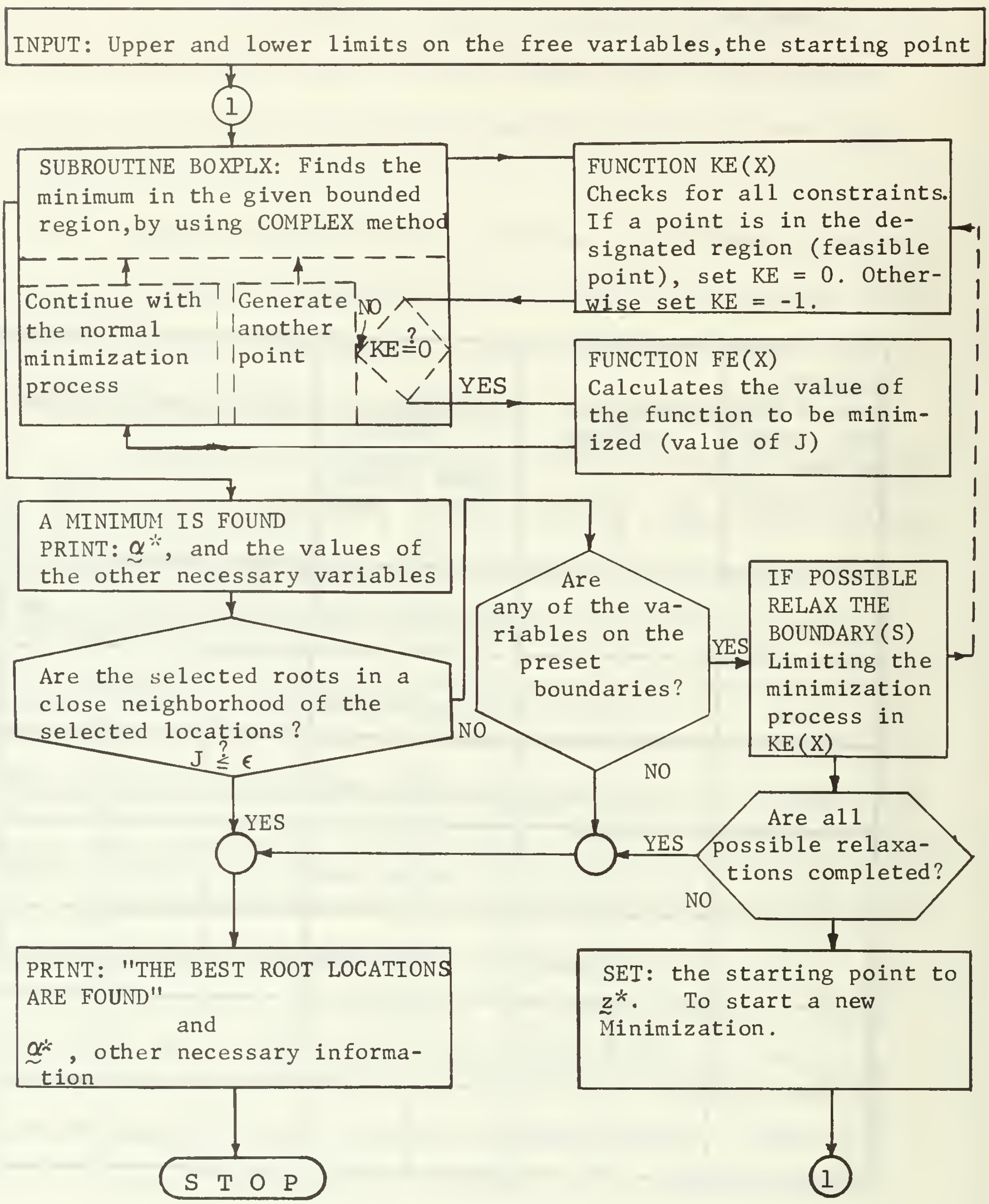
10. <u>Interpret the Results</u>

At the first minimization step the undesired roots were kept to the left of $s = -\sigma_{1t} = -1.5$ line. The desired roots were probably dominant (since the transient due to the undesired real root settles down in $1/6^{th}$ of the time of the transients caused by complex roots), but the dominant roots were not close to the desired locations and the transient response would not be satisfactory. At the last step of the relaxation the desired roots were exactly located but their dominance was questionable since an undesired real root was located close to the desired roots.

To make the final decision on the parameter values, residues can be calculated; this would certainly remove the doubts about the dominance of the desired roots, but since a digital computer is assumed to be available and digital simulation of the system avoids the lengthy residue calculations, it is best to obtain the time response of the system and compare it with the desired second-order response.

LIMITING VALUE FOR THE UNDESIRED ROOTS ($-\alpha_{lt}$)	UNDESIRED ROOTS		DESIRED ROOTS		SYSTEM PARAMETERS			$J^{\frac{1}{2}}$
	#1	#2	Real Part	Imag. Part	k	p	K	
-1.5	-1.5	-27.15	-0.249	±0.496	0.24	27.15	0.46	0.25
-1.4	-1.4	-27.56	-0.297	±0.469	0.1	27.55	0.43	0.205
-1.3	-1.3	- 9.85	-0.336	±0.492	0.17	9.83	0.463	0.165
-1.2	-1.2	-13.21	-0.386	±0.494	0.1	13.18	0.473	0.114
-1.1	-1.1	-11.23	-0.429	±0.499	0.1	11.19	0.479	0.07
-1.0	-1.0	- 9.733	-0.473	±0.497	0.1	9.68	0.474	0.02
-0.9	-0.94	- 9.525	-0.5	±0.5	0.1	9.467	0.474	0.00008

TABLE II.1



ϵ is the predetermined positive constant (Stopping criterion)

Fig. II.5. Computational flow chart for the method "Optimization for the Best Root Locations in the s-Domain".

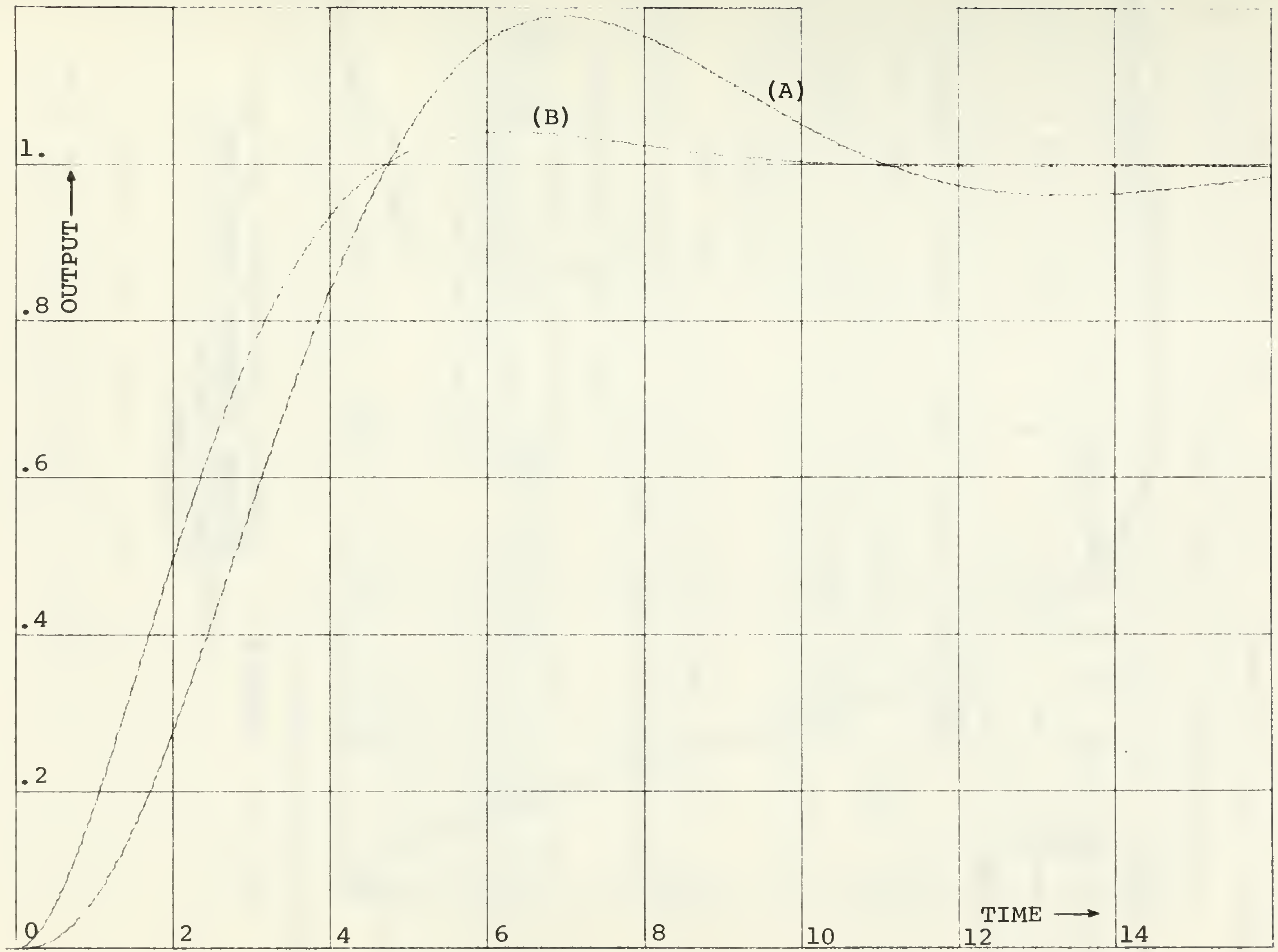


Fig.II.6. EXAMPLE 1. System's time response (A) and desired response(B) to a unit step input. Parameter values listed on the first line of TABLE II.1 are used.

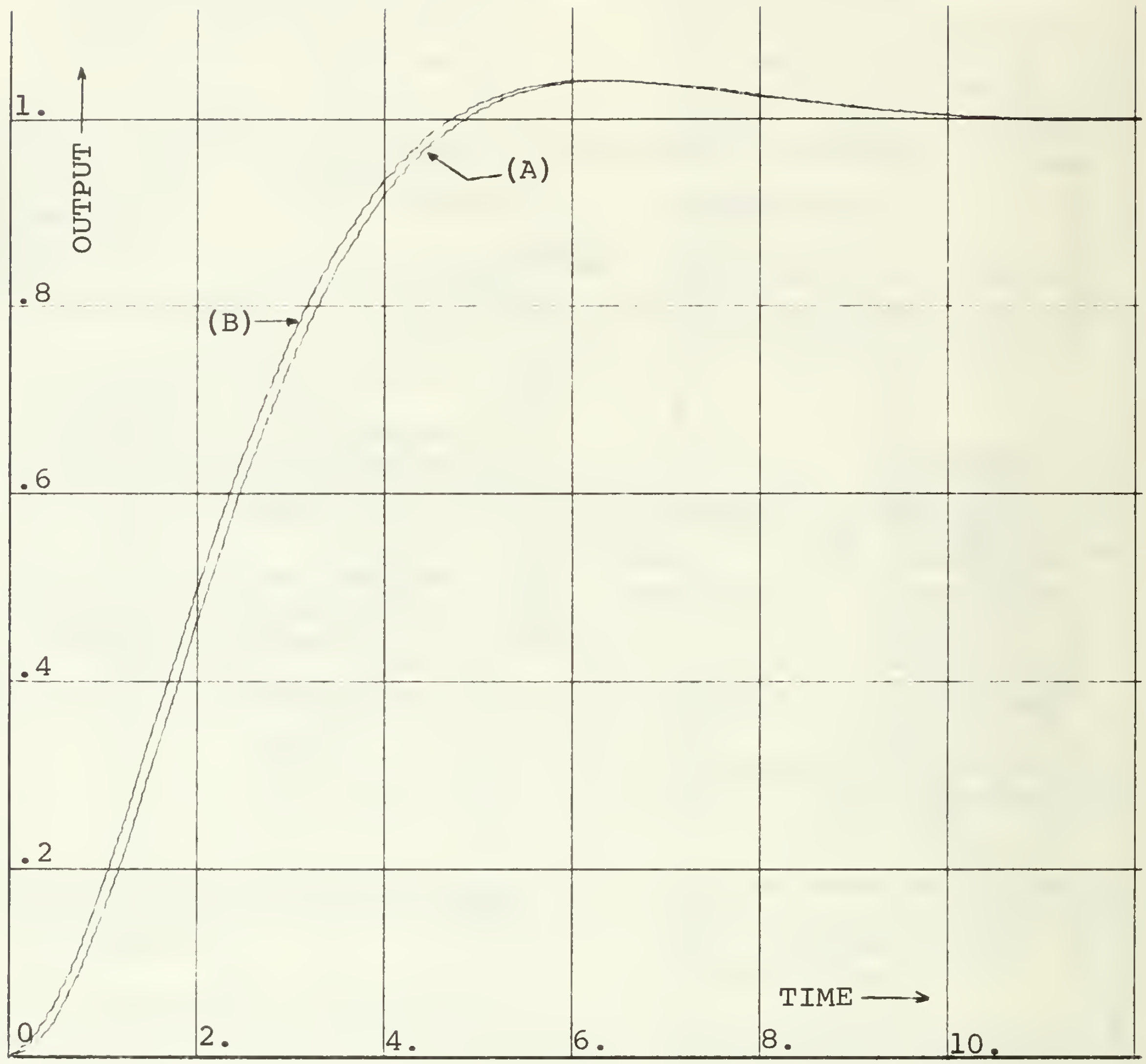
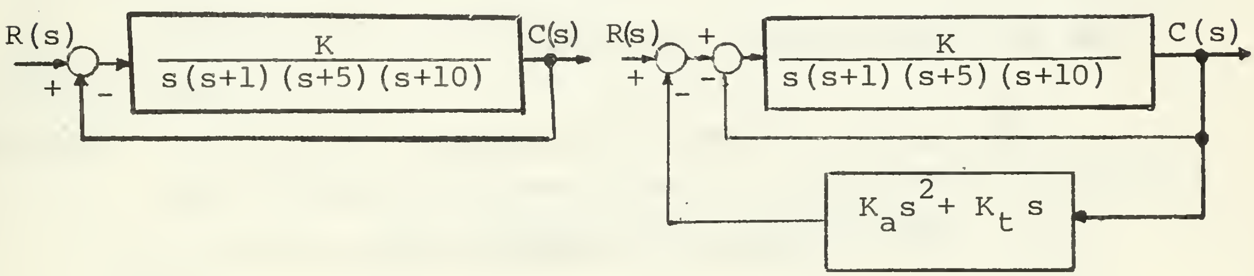


Fig.II.7. EXAMPLE 1. System's time response (A) with optimum parameter values and desired response (B) to a unit step input.

Fig. II.6 is the response (to a unit step input) for the first line of Table II.1; the desired second-order response is also shown. Fig. II.7 shows the time response of the system (to a unit step input) for the parameter values tabulated as the last line of Table II.1, where the complex roots were exactly located at the desired locations ($\sigma = \sigma_d = 0.5$ and $\omega = \omega_d = \pm 0.5$) but a real root was close to the desired roots (at $s = -0.94$). Since the compensated system's response is very close to the desired response, the optimum parameter values tabulated as the last line of Table II.1 yield a very satisfactory design.

H. EXAMPLE 2

The linear fourth-order system shown in Fig. II8(a) is given. Steady-state performance specifications require the system gain to be equal to



(a) The uncompensated system (b) The compensated system

Fig. II.8. EXAMPLE 2

(or greater than) 1000 ($K \geq 1000$), but the root-locus study shows that the system is unstable if $K > 193.23$.

1. The dynamic performance specifications[1] can be met by locating a pair of complex (dominant) roots at $\sigma_d = 1$ and $\omega_d = \pm 2$.

2. The desired root locations are not on the uncompensated system's root locus, and the system gain is much less than the desired value (for the part of the root locus in the left hand side of the s-plane); therefore, some kind of compensation is needed.

3. A combination of tachometer and acceleration feedback will be used to compensate the system. The configuration of the compensated system is shown in Fig. II8(b).

4. The transmission function of the compensated system is

$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{s^4 + 16s^3 + (65+KK_a)s^2 + (50+KK_t)s + K}, \qquad (II.32)$$

and the characteristic equation is

$$s^4 + 16s^3 + (65+KK_a)s^2 + (50+KK_t)s + K = 0 . \qquad (II.33)$$

5. The desired-roots quadratic is

$$P_d(\sigma, \omega, s) = s^2 + 2\sigma s + c \qquad (II.34)$$

where

$$c \triangleq \sigma^2 + \omega^2 .$$

---

[1] Paragraph numbers indicate the corresponding steps in EXAMPLE 1.

The undesired-roots polynomial is

$$P_{ud}(\underset{\sim}{\beta}, s) = s^2 + 2as + b \tag{II.35}$$

where

$$\underset{\sim}{\beta} = (a \; b)^T .$$

Eq. (II.6) is used to factor the system's characteristic equation,

$$\begin{aligned} &s^4 + 16s^3 + (65+KK_a)s^2 + (50+KK_t)s + K \equiv \\ &\quad s^4 + 2(a+\sigma)s^3 + (b+c+4a\sigma)s^2 + 2(bc+ac)s + bc. \end{aligned} \tag{II.36}$$

By equating the coefficients of the corresponding terms of identity (II.36),

$$\begin{aligned} &a + \sigma = 8 \\ &b + c + 4a\sigma = 65 + KK_a \\ &2(b\sigma + ac) = 50 + KK_t \\ &bc = K \end{aligned} \tag{II.37}$$

The $\underset{\sim}{z}$-vector, representing the independent variables in the minimization process, is chosen to be

$$\underset{\sim}{z} = (\sigma \; c \; K)^T . \tag{II.38}$$

The constraining equations are

$$\begin{aligned} &a = 8 - \sigma \\ &b = K/c \\ &K_a = (b + c + 4a\sigma - 65)/K \\ &K_t = (2\,(b\sigma + ac) - 50)/K \end{aligned} \tag{II.39}$$

6. All the variables must be positive (or zero), and the other constraints are

$$K \geq 1000$$

$$0 \leq K_a \leq 10$$

$$0 \leq K_t \leq 10 \qquad \text{(II.40)}$$

$$\sigma^2 - c < 0$$

$$a - (a^2 - b)^{\frac{1}{2}} \geq \sigma_{1t} = 6 \qquad \text{(II.40a)}$$

7. The performance index is

$$J = (1.-\sigma)^2 + (2.-\omega)^2 \qquad \text{(II.41)}$$

8. SUBROUTINE BOXPLX was again used for the minimization.

9. None of the variables was on the limiting boundaries; therefore, no relaxation was necessary. The result of the minimization is tabulated in Table II.2.

10. The compensated system's response (to a unit step input) and the desired response are shown in Fig. II.9.

LIMITING VALUE FOR THE UNDESIRED ROOTS	UNDESIRED ROOTS		DESIRED ROOTS		SYSTEM PARAMETERS		
$(-\sigma_{1t})$	Real part	Imag. part	Real part	Imag. part	K^*	K_a^*	K_t^*
-6	-6.999	-199.585	-1.001	±2.002	1000	.1676	.4197

TABLE II.2

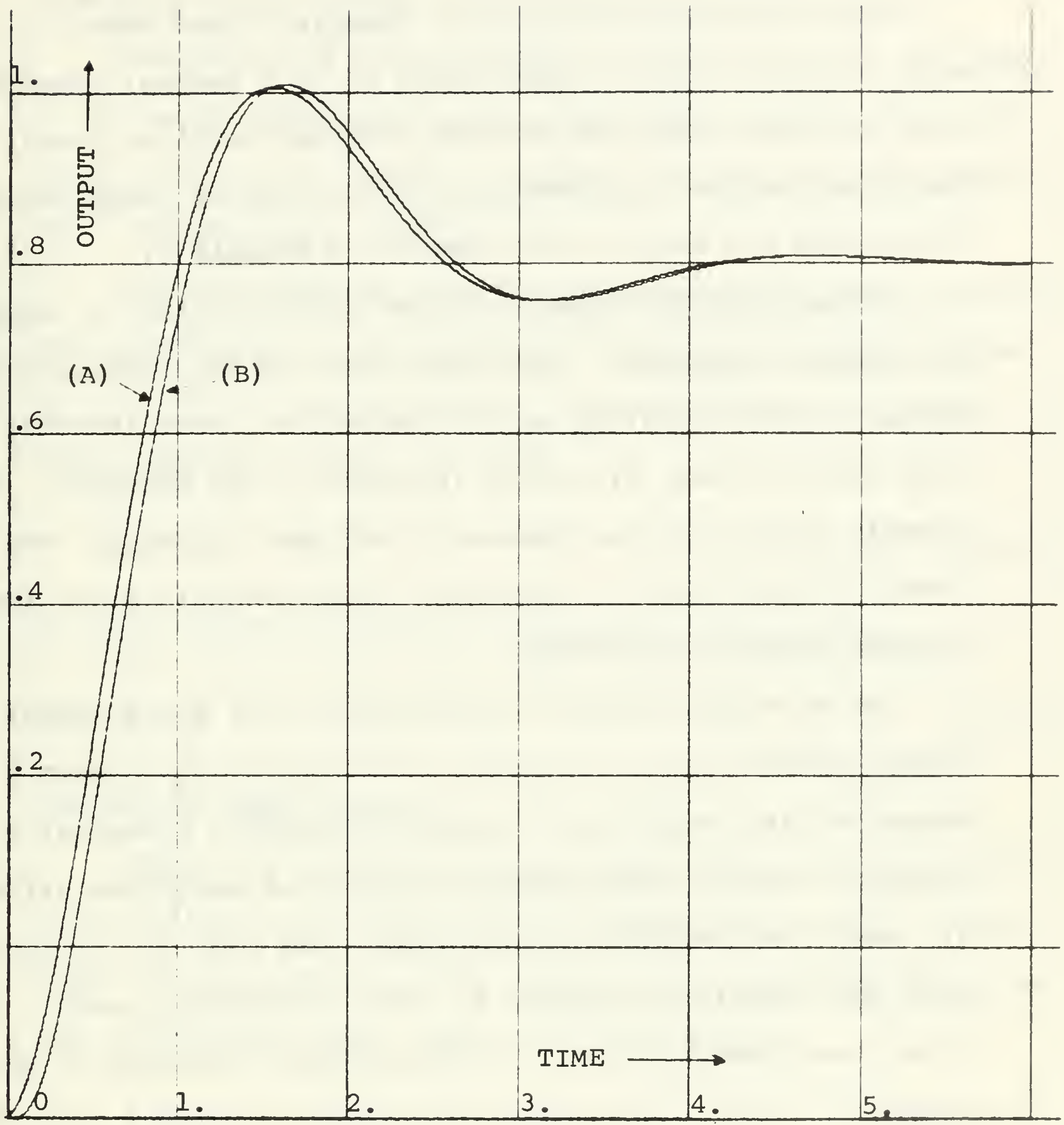


Fig.II.9. EXAMPLE 2. Time response (A) of optimally compensated system and the desired response (B) (to a unit step input).

I. EXAMPLE 3

The linear system given in Example 2 (and shown in Fig. II.8(a)) will be compensated for the fastest response (the shortest rise time and the shortest settling time). The steady-state requirements and the type of compensation to be used are exactly the same as in Example 2.

Since the only dynamic system specification is "obtain the fastest response", the actual form of the transient response and the location of the dominating roots (whether they exist or not) are of no interest to the designer. This example shows that the dominant-root idea, although very useful for many design problems, is not mandatory for the system design procedures.

As mentioned earlier (in Section II.A) the s-domain design methods try to relocate the roots of the system's characteristic equation to obtain a suitable transient response. Most of these methods ignore the magnitude effects (to avoid the residue calculations), and they try to classify the transient response as "the fast-varying part", and "the slowly-varying part". This procedure depends on the assumption that, "magnitude-wise all of the roots have the same importance in the transient response".

With the same assumption one may try to obtain "the fastest response" by making the real parts of all the roots as large as possible, because the rise time and the settling time of a transient response caused by a root is inversely proportional to the real part of that root.

The solution of the problem contains exactly the same steps as the previous example (see Eqs. (II.32) - (II.41)). Only the last two equations should be changed, specifically Eqs. (II.40a) and (II.41).

Eq. (II.40a) was a constraining equation which ensured the confinement of the undesired roots to the left of the $s = -\sigma_{1t}$ line for the case of two real undesired roots. In this problem it takes the form

$$a - (a^2 - b)^{\frac{1}{2}} \geq \sigma \ . \qquad \text{(II.42)}$$

If the undesired roots are complex[1], the equation

$$a \geq \sigma \qquad \text{(II.43)}$$

serves the same purpose.

These two constraining equations ensure that the magnitudes of the real parts of the undesired roots are greater than the magnitudes of the real parts of the desired roots. The performance index takes the simple form

$$J = -\sigma \quad , \quad (\sigma \geq 0) \ , \qquad \text{(II.44)}$$

because it is only necessary to maximize the real parts of the desired roots.

---

[1] For this problem all roots are undesirable, but just for purposes of terminology the names used in Example 2 are retained. The roots of the polynomial P_{ud} are still called "the undesired roots" and the roots of P_d are called "the desired roots".

The results of the minimization are shown in Table II.3, and the time response to the unit step input is shown in Fig. II.10. The desired response for Example 2 is also plotted in the same figure for easy comparison.

UNDESIRED ROOTS		DESIRED ROOTS		SYSTEM PARAMETERS		
Real part	Imag. part	Real part	Imag. part	K^*	K_a^*	K_t^*
-4.	±34.8	-4.	±3.57	1000.	.0625	.458

TABLE II.3

J. EXAMPLE 4

In some design problems it may be necessary to insert a lag compensator which has a dipole (a pole and a zero) very close to the origin. In this case some undesired root(s) may be located closer to the origin than the desired roots. Problems of this type may be solved by using the normal step-by-step procedure given in Example 1; if this is done, an acceptable answer may be obtained by relaxing the limiting value for the undesired roots (σ_{1t}) by a considerable amount toward the origin (as in step 9 of Example 1). If the designer can gain enough insight to the problem (probably at step 2 of the normal step-by-step procedure) he may directly select $\sigma_{1t} = 0$ (to ensure the stability of the system) and save some computer time by avoiding the relaxation steps for σ_{1t}.

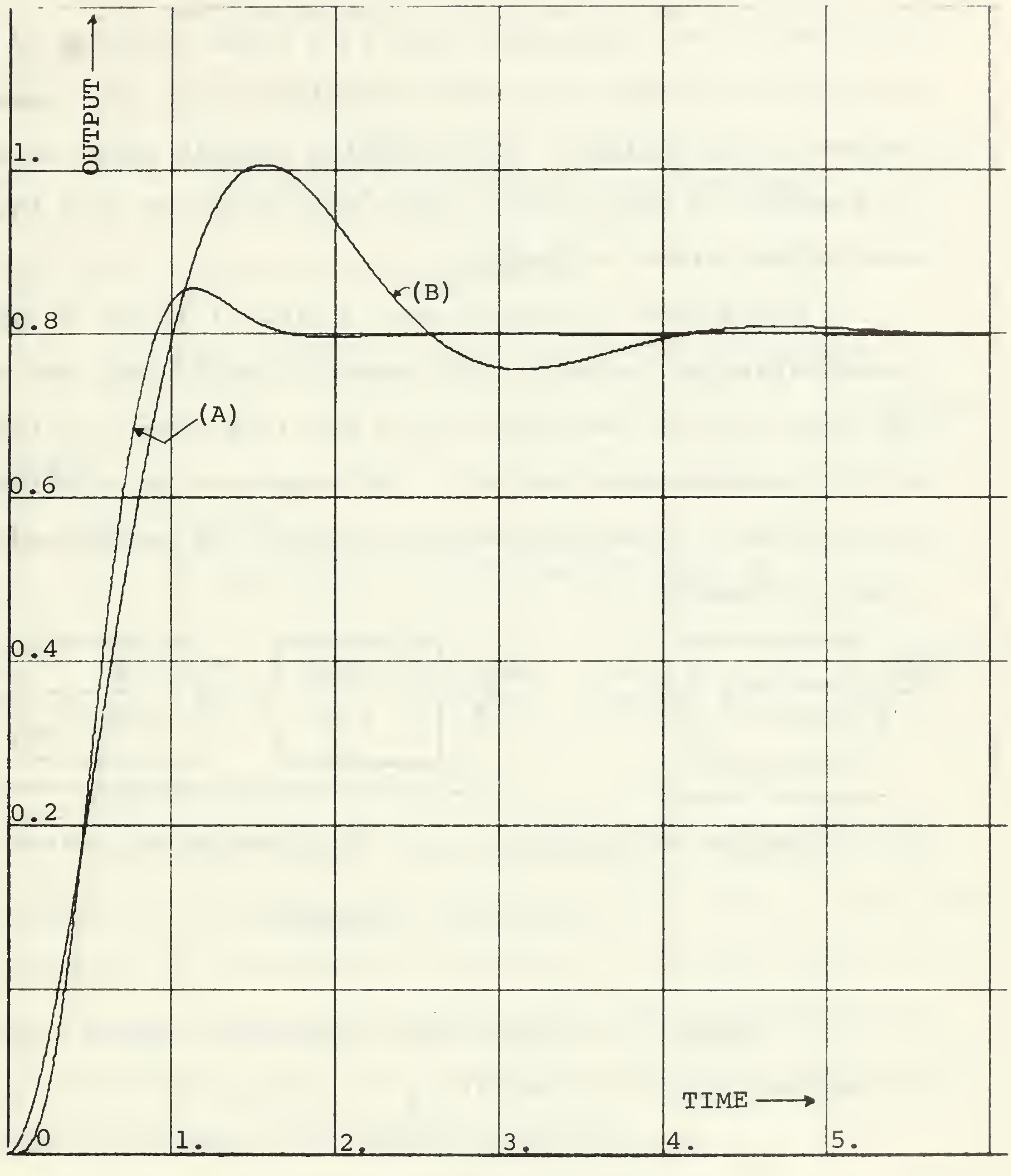
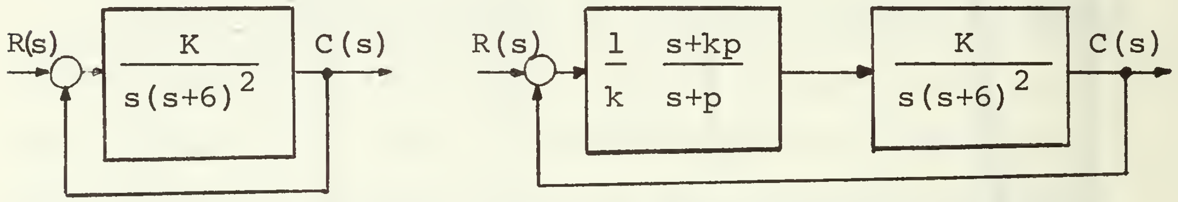


Fig. II.10. EXAMPLE 3. The fastest time response (A) and the desired response (B) for EXAMPLE 2.

In another approach the designer may identify and confine some of the undesired roots to a close vicinity of the origin while keeping the other undesired roots in a remote region of the s-plane. In the design example given below this method is used just to explain a variation from the main method given in Example 1.

A third-order system (shown in Fig. II.11(a)) is given. Steady-state performance requirements specify that the system gain (K) must be greater than 425 (the stability limit of the uncompensated system). The step-by-step solution is given below; paragraph numbers indicate the corresponding steps in Example 1.



(a) Uncompensated system (b) Compensated system

Fig.II.11. EXAMPLE 4

1. Dynamic performance specifications require a pair of dominant roots located at $\sigma_d = 1$, and $\omega_d = \pm 2$.

2. - 3. The designer decides on a cascade lag compensator by sketching the root locus of the uncompensated system. The compensated system is shown in Fig. II.11(b).

4. The transmission function of the compensated system is

$$T(s) = \frac{(K/k)(s+kp)}{s^4+(12+p)s^3+(36+12p)s^2+(36p+K/k)s+Kp}, \qquad (II.45)$$

and the characteristic equation is

$$P_{ch}(\underset{\sim}{\alpha},s)=s^4+(12+p)s^3+(36+12p)s^2+(36p+K/k)s+Kp = 0 \qquad (II.46)$$

where

$$\underset{\sim}{\alpha} = (k\ p\ K)^T .$$

5. The quadratic for the desired root is

$$P_d(\sigma,\omega,s) = s^2 + 2\sigma\, s + c \qquad (II.47)$$

where

$$c \triangleq \sigma^2 + \omega^2 .$$

For this problem the characteristic equation of the system cannot have more than two complex roots; since the designer is interested in the part of the root locus where the desired roots are complex, then the undesired roots must be real. For this reason the polynomial representing the undesired roots can be written as

$$P_{ud}(\underset{\sim}{\beta},s) = (s+a)\cdot(s+b) = s^2+ (a+b)s + ab \qquad (II.48)$$

The characteristic equation is factored by using identity (II.6)

$$P_{ch} \equiv P_d\, P_{ud} \qquad (II.6)$$

$$s^4+(12+p)s^3+(36+12p)s^2+(36p+K/k)s+Kp \equiv \tag{II.49}$$

$$s^4+(a+b+2\sigma)s^3+(ab+c+2a\sigma+2b\sigma)s^2+(2ab\sigma+ac+bc)s+abc,$$

and σ, p, and b are chosen as the variables in the minimization process; in other words,

$$\underset{\sim}{z} = (\sigma \; p \; b)^T \tag{II.50}$$

The constraining equations can be written (from Eq. (II.49)) as

$$\begin{aligned} a &= 12 + p - b - 2\sigma \\ c &= 36 + 12p - 2\sigma(a+b) - ab \\ K &= abc/p \\ k &= K/(-36p + 2ab\sigma + c(a+b)). \end{aligned} \tag{II.51}$$

6. All of the variables must be greater than or equal to zero, and the other constraining inequalities are

$$\begin{aligned} &K \geq 425 \\ &1 < k \leq 10 \\ &b > \sigma_{1t-1} \quad \text{limit for the remote undesired root} \\ &0 < a \leq \sigma_{1t-2} \text{ limits for the close undesired root.} \end{aligned} \tag{II.52}$$

7. The performance index

$$J = (1.-\sigma)^2 + (2.-\omega)^2 \tag{II.53}$$

is used for minimization by SUBROUTINE BOXPLX.

9. Some suitable logic to relax the boundaries was inserted in the main program. The remote undesired root (at $s = -b$) is confined to the left of the $s = -\sigma_{1t-1}$ line and the close undesired root (at $s = -a$) is confined between the $s = 0$ and $s = -\sigma_{1t-2}$ lines, where $\sigma_{1t-1} = 8$, and $\sigma_{1t-2} = 1.5$. The first limiting line would be shifted to the right and the second one shifted to the left, but no relaxation is done because the minimum is not on the enclosing boundaries. The result of the minimization is tabulated as Table II.4.

10. The compensated system's response (to a unit step input) and the desired response are shown in Fig. II.12. Although the desired roots are located exactly at the selected locations, they are obviously not dominant becasue the two time responses are not close to each other.

LIMITING VALUES FOR THE UNDESIRED ROOTS		UNDESIRED ROOTS		DESIRED ROOTS		SYSTEM PARAMETERS		
$-\sigma_{1t-1}$	$-\sigma_{1t-2}$	Close	Remote	Real part	Imag. part	K	p	k
-8.	-1.5	-1.38	-8.723	1.	±2.	563.	.107	7.952

TABLE II.4

K. CONCLUDING REMARKS

The method "Optimization for the Best Root Locations In the s-Domain" provides fast, direct and efficient solution to the linear system design problems of classical control.

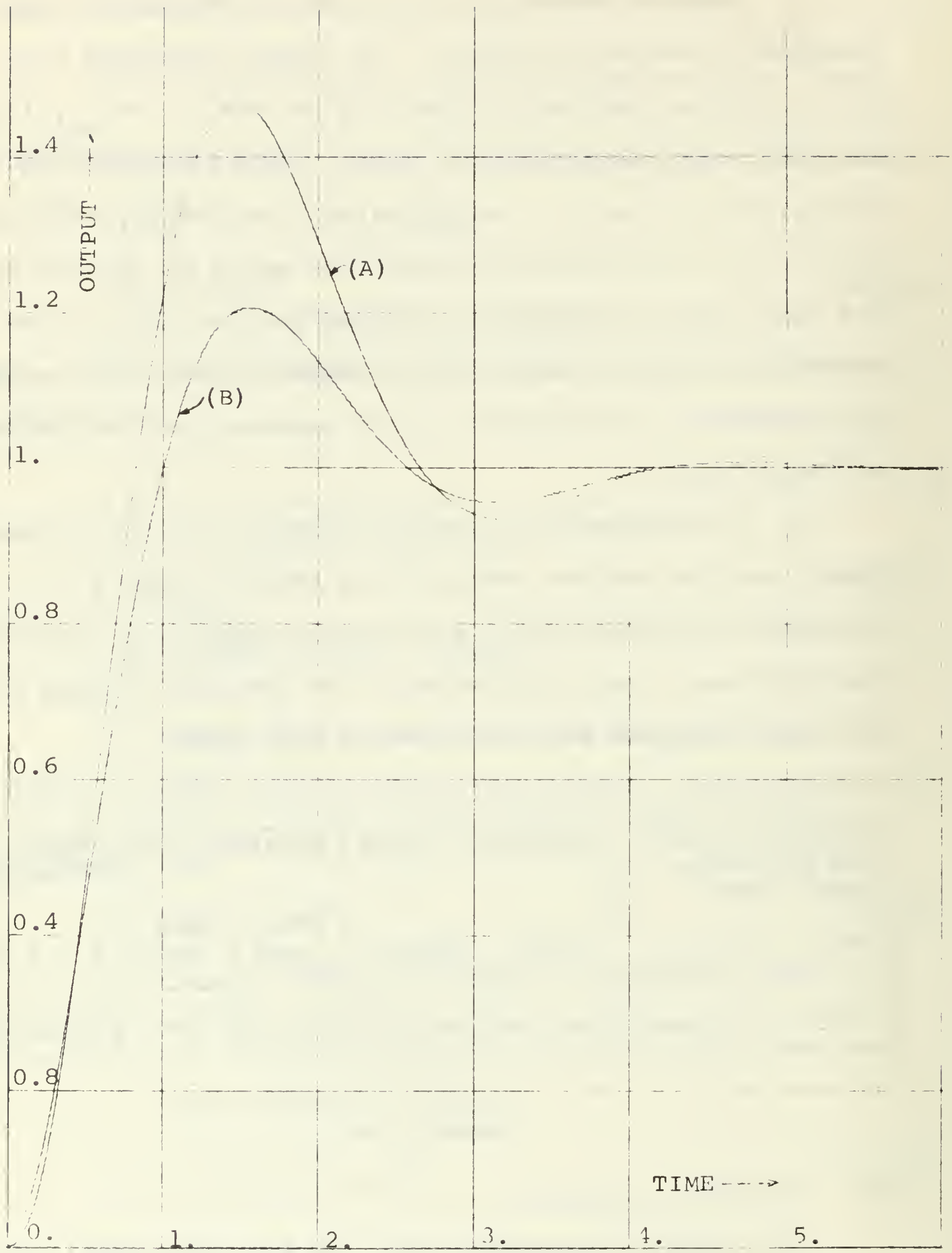


Fig. II.12. EXAMPLE 4. Time response (A) of the compensated system with a lag compensator and the desired response (B) to a unit step input.

Systems with more than two free parameters can be handled easily.

For high-order systems, algebraic manipulations necessary to factor the characteristic equation may get too involved. If the number of free parameters is very small compared to the order of the system, these algebraic manipulations may become prohibitively time consuming.

III. OPTIMIZATION FOR THE BEST RESPONSE IN THE TIME-DOMAIN -- APPLICATIONS TO LINEAR SYSTEM DESIGN PROBLEMS

A. GENERAL

Most of the methods for the design of dynamic systems are inherently indirect, s-domain methods, for example, try to adjust the root locations of the system characteristic equation to obtain a suitable transient response. For a high-order system it may be very difficult to decide on a set of suitable root locations which defines a desirable transient response; for this reason, the final step of an s-domain design procedure is usually the simulation of the system, to check if the new root locations yield an acceptable transient response.

The method presented in this chapter eliminates the intermediate steps and directly tries to shape the transient response of the system. This method is called "Optimization for the Best Response in the Time-Domain".

The first step of the method is selecting a "MODEL RESPONSE"[1] by interpreting the dynamic performance specifications. Then the state equations of the system are integrated with respect to time with a given input

---

[1] The word "response" implies "time response" even when it is used alone.

function, and the output of the system is obtained and sampled at suitable time intervals. These samples are compared with the desired values of the response (defined by the model response). A performance index, which measures the deviation of the sample values from the desired values (from zero time to the expected end of the settling period of the system) is defined. Minimization of this performance index with respect to the free system parameters yields the best time response of the system, for the given specifications and the system configuration.

In this chapter, the simple philosophy of the method is described and the method is applied to the linear system design problems which were solved in Chapter II by using the method "Optimization for the Best Root Locations in the s-Domain".

B. SELECTING THE MODEL RESPONSE

Selection of the model response is determined by the dynamic performance specifications. If the system is to be operated always with a specific input signal, then this time function (input) can also be used as the model response for the comparison. A step input (test signal) is very common in dynamic system design procedures. The same step function may also be used as the model response; if this is done, however, it may be necessary to insert some constraints on the states of the system to fulfill other dynamic performance specifications, such as peak overshoot, etc. (see Examples 7 and 9).

If the dynamic performance specifications are too tight (that is, if expected rise time, peak overshoot, settling time, frequency of oscillations are given within some small tolerance limits) a second-order model system may be used to represent the desired model response easily and efficiently. The output of the model system is obtained by integrating the state equations of the model. Using a second-order system's output as the model response is, of course, equivalent to the dominant-roots concept in the s-domain.

C. CONSTRAINTS

The constraints on the free system parameters are usually linear or can be approximated by linear inequalities -- this simplifies the computer programming part of the design procedure.

There may be some constraints on the state of the system due to inherent nonlinear characteristics of the system components. These constraints can be represented by linear or nonlinear inequalities and they do not introduce much difficulty to the design procedure (see Chapter IV).

To obtain some design characteristics, additional constraints on the states of the system may be introduced by the designer. For example, in one design procedure the peak overshoot of the system can be kept below a certain level by inserting suitable constraints on the output state. In this thesis, penalty functions are employed to handle this type of constraint (see Examples 7 and 9).

D. DEVELOPMENT OF THE METHOD

The state equations of an n^{th}-order, single-input system can be written in the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{b}r(t) \tag{III.1}$$

where $\underset{\sim}{x}(t)$ is the state n-vector and the "dot" over a variable is used to indicate the time derivative (d/dt). $\underset{\sim}{A}$ is an n x n matrix, r(t) is the scalar time function representing the input to the system and $\underset{\sim}{b}$ is an n-vector.

The output equation is

$$c_s(t) = \underset{\sim}{a}^T \underset{\sim}{x}(t) + h_o\, r(t) \tag{III.2}$$

where $c_s(t)$ is the output of the system, $\underset{\sim}{a}$ is an n-vector and h_o is a scalar.

Assume that the design is to be carried out by using a second-order model response. The state equations for the second-order model system[1] can be written as

$$\dot{\underset{\sim}{y}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_{n_d}^2 & -2\zeta_d\omega_{n_d} \end{bmatrix} \cdot \underset{\sim}{y}(t) + \begin{bmatrix} 0 \\ \omega_{n_d}^2 \end{bmatrix} \cdot r(t) \tag{III.3}$$

---

[1]This step may take different forms depending on the choice of the model response. If the input time function (r(t)) is to be used as the model response, for example, no extra effort is necessary to generate the model response.

where $\underset{\sim}{y}(t)$ is the state vector of the model system, ζ_d represents the desired damping ratio and ω_{n_d} represents the desired natural frequency of the model system. The output of the model is

$$c_m(t) = y_1(t) \ . \tag{III.4}$$

A performance index which is suitable for many design problems is

$$J(\underset{\sim}{\alpha}) = \int_0^{t_f} (c_s(t) - c_m(t))^2 dt \ , \tag{III.5}$$

where $\underset{\sim}{\alpha}$ is an N_c-dimensional vector representing the free system parameters, and $c_m(t)$ represents the model response (for the case under discussion it is the output of the second-order model). t_f (the final time of the integration) can be taken as the settling time of the second-order model; it can be calculated by using the equation

$$t_f = \frac{4}{\zeta_d \omega_{n_d}} = \frac{4}{\sigma_d} \tag{III.6}$$

where σ_d is the real part of the desired root locations.

If another model response is used, the final time can be guessed by using the performance specifications -- it should be long enough to cover at least one period of the lowest frequency of the transients which occur during the minimization process. The value of the performance index depends on t_f, but once it is chosen (long enough) and fixed, it does not affect the optimum parameter values obtained at the end of the minimization. If t_f is chosen longer than is necessary, extra computer time may be spent. For this reason, if the designer does not have any idea about the dynamic behavior of the system, he should simulate the system by using the starting values of the free system parameters as the first step of the design procedure.

The performance index in the sampled form is written as

$$J(\underset{\sim}{\alpha}) = \sum_{k=0}^{n_f} (c_s(kT) - c_m(kT))^2 \tag{III.7}$$

where n_f is a positive integer defined as

$$n_f = \frac{\text{final time of the calculations}}{\text{Sampling period}} = \frac{t_f}{T} . \tag{III.8}$$

When J is minimized with respect to the free system parameters ($\underset{\sim}{\alpha}$) in the bounded region defined by the constraints (see Section III.C), the optimum parameter values ($\underset{\sim}{\alpha}^*$) are found.

The form of the performance index can be modified to fit the needs of the specific design problem; variation from the basic form (Eqs. (III.5) and (III.7) are employed in the examples given in the later parts of the thesis (see particularly Examples 7 and 9).

E. EXAMPLE 5

The linear system design problem solved as Example 1 in Chapter II is reconsidered here. This time, a solution is obtained by using the method "Optimization for the Best Response in the Time-Domain". The system (shown in Fig. II.3(a)) and all specifications are kept the same to provide a basis for comparison. A step-by-step solution is given below:

1. The Model Response

The desired dynamic response was represented by two complex roots, in Example 1; the desired dynamic response (the model response) is the output of a second-order model system defined by dominant roots located at $\sigma_d = 0.5$, $\omega_d = \pm 0.5$. The state equations of the second-order model are written from Eq. (III.3)

$$\begin{aligned} \dot{x}_5(t) &= x_6(t) \\ \dot{x}_6(t) &= 0.5(r(t) - x_5(t)) - x_6(t) , \end{aligned} \tag{III.10}$$

where $x_5(t) = y_1(t)$ and $x_6(t) = y_2(t)$ are used for programming convenience. A unit step input is assumed ($r(t) = \mathbb{1}(t)$). The model response is

$$c_m(t) = x_5(t) . \tag{III.11}$$

2. Type of Compensation

The same cascade compensator (shown in Fig.II.4(a)) is used for compensation.

3. <u>State Equations of the System</u>

The state equations of the system are written from the transmission function (Eq. (II.11) in the rational canonical form

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= x_4(t) + h_3\ r(t)\\ \dot{x}_4(t) &= -b_o x_1(t) - b_1 x_2(t) - b_2 x_3(t) - b_3 x_4(t) + h_4 r(t)\end{aligned} \tag{III.12}$$

where

$$\begin{aligned}b_o &= Kp & b_3 &= 2 + p\\ b_1 &= p + K/k & h_3 &= K/k\\ b_2 &= 1 + 2p & h_4 &= b_o - h_3 b_3\ .\end{aligned} \tag{III.13}$$

The output of the system is

$$c_s(t) = x_1(t)\ . \tag{III.14}$$

4. <u>The Performance Index</u>

The performance index defined in Eq. (III.5) is suitable for this design problem, hence

$$J \triangleq x_7(t) = \int_o^{t_f} (x_1(t) - x_5(t))^2 dt\ , \tag{III.15}$$

and its time derivative is

$$\dot{x}_7(t) = (x_1(t) - x_5(t))^2\ . \tag{III.16}$$

The final time for the integration is taken as the settling time of the model response

$$t_f = 4/\sigma_d = 8 \text{ seconds.} \tag{III.17}$$

5. Constraints

The only constraining inequalities are

$$\begin{aligned} &0.1 \le k \le 10 \\ &p > 0 \\ &0.4 \le K \le 0.5 \; . \end{aligned} \tag{III.18}$$

6. Minimization

The performance index ($x_7(t)$) is minimized with respect to the free system parameters ($\underset{\sim}{\alpha}$). A gradient method was used for the minimization (COMPUTER PROGRAM II, given at the end of the thesis is typical for this method). At each step of the minimization procedure, Eqs. (III.10), (III.12) and (III.16) are integrated from $t = 0$ to t_f by using a fourth-order Runge-Kutta method[1]. The optimum parameter values are $k^* = 0.1$, $p^* = 9.84$, and $K^* = 0.4945$. The optimum time response of the system and the desired response are shown in Fig. III.1.

The system's response shown in this figure (and its closeness to the desired response) should be compared with the response shown in Fig. II.7.

---

[1] SUBROUTINE RKLDEQ, N.P.G.S. Computer Facitlity Program Library.

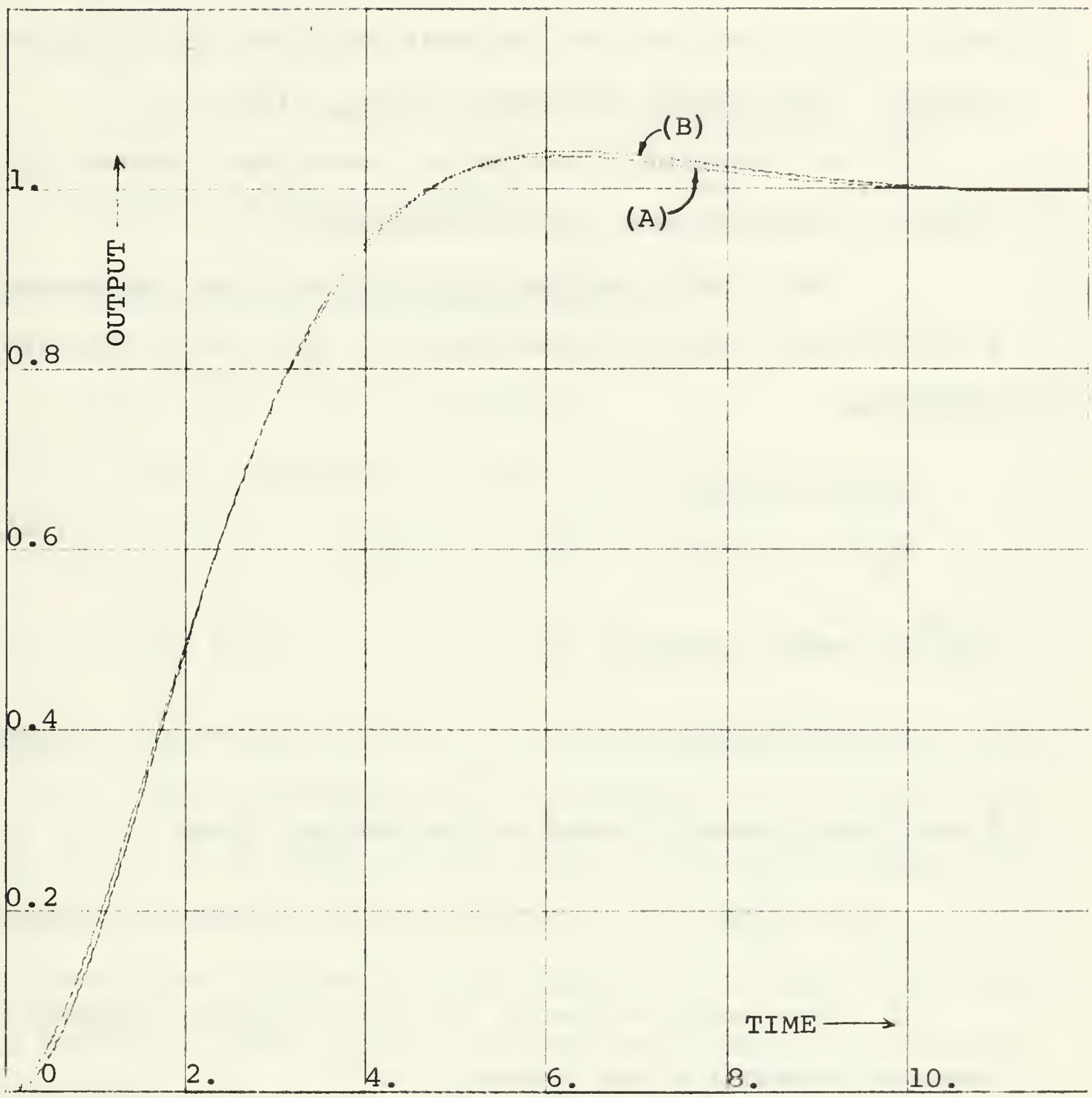


Fig. III.1. EXAMPLE 5. Compensated system's response (A) and the desired response (B) to a unit step input.

F. EXAMPLE 6

Example 2 of Chapter II is re-solved by using the method "Optimization for the Best Response in the Time-Domain". The system is shown in Fig. II.8.

In the solution given below, paragraph numbers indicate the corresponding steps in Example 5.

1. The model response is obtained from the output of a second-order model system which is defined by the state equations

$$\begin{aligned} \dot{x}_5(t) &= x_6(t) \\ \dot{x}_6(t) &= 5(r(t) - x_5(t) - 2x_6(t) \ , \end{aligned} \tag{III.19}$$

and the model response is

$$c_m(t) = x_5(t) \ . \tag{III.20}$$

A unit step input is used in the design, thus

$$r(t) = \mathbb{1}(t) \ . \tag{III.21}$$

2. The same tachometer and acceleration feedbacks are used to compensate the system.

3. The state equations of the system are written directly from Fig. II.8(b).

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= [r(t)-x_1(t)-K_t x_2(t)-K_a x_3(t)]K-[50x_2(t)+65x_3(t)+16x_4(t)] \end{aligned} \tag{III.22}$$

The output of the system is

$$c_s(t) = x_1(t) \ . \qquad \text{(III.23)}$$

4. The time derivative of the performance index is

$$\dot{x}_7(t) = (x_1(t) - x_5(t))^2 \ . \qquad \text{(III.24)}$$

The final time for the integration is

$$t_f = 4/\sigma_d = 4/1 = 4 \text{ seconds.} \qquad \text{(III.25)}$$

5. The constraints on the free variables are

$$\begin{aligned} 0 &\le K_t \le 10 \ , \\ 0 &\le K_a \le 10 \ , \end{aligned} \qquad \text{(III.26)}$$

and the system gain is set to its desired value ($K = 1000$).

6. The performance index defined by the integral of Eq. (III.24) is minimized by using a gradient method; the optimum parameter values are $K_a^* = 0.142$, $K_t^* = 0.4$, and the optimum time response of the system (for a unit step input) is plotted in Fig. III.2. In the same figure the desired response is also shown. The result of s-domain synthesis procedure was given in Fig. II.9.

G. EXAMPLE 7

The linear system design problem solved in Examples 2, 3, and 6 is re-considered here. As in Example 3 of Chapter II dynamic performance specifications require "the

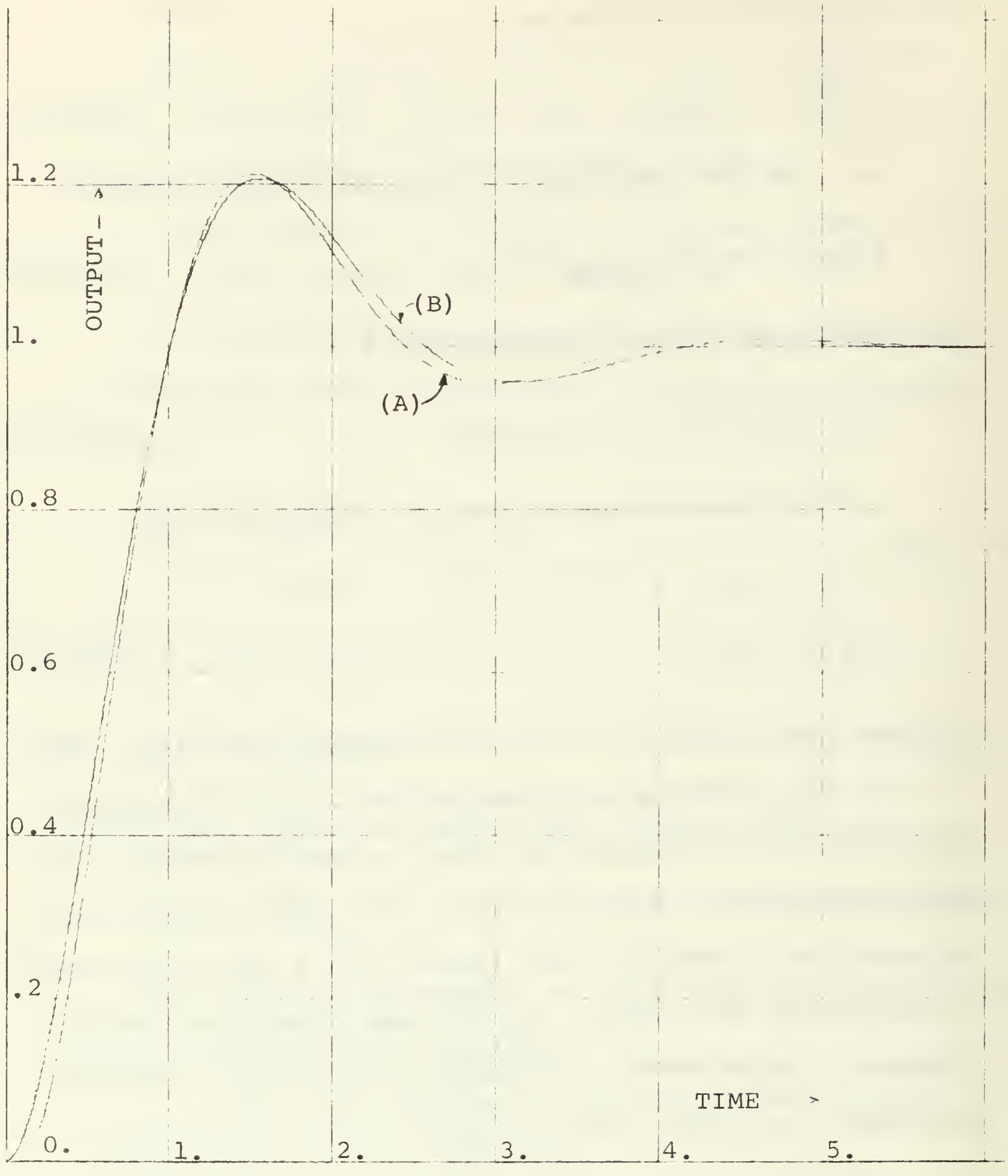


Fig. III.2. EXAMPLE 6. Compensated system's response (A) and the desired response (B) to a unit step input.

fastest response" to a unit step input. This problem is solved three times by using slightly different techniques to demonstrate the effects of the different forms of performance indices.

Solution 1

1. The unit step input is used as the model response:

$$c_m(t) = r(t) = \mathbb{1}(t) \quad . \tag{III.27}$$

Other parts of the solution are exactly the same as the one given in the previous example except that the performance index now takes the form

$$\dot{x}_7(t) = (1.-x_1(t))^2 \tag{III.28}$$

2. The new performance index is minimized with respect to the free system parameters (K_a and K_t) ant the optimum parameter values are $K_a^* = 0.0296$, $K_t^* = 0.4546$. The optimum response is plotted in Fig. III.3 . In the same figure the desired response of Example 6 is also shown, for comparison.

Solution 2

The optimum parameter values in Solution 1 yield a faster response (shorter rise time and shorter settling time) than obtained in Example 6, but the response is somewhat oscillatory and the settling time is about four seconds. If dynamic performance specifications require a less oscillatory output and a settling time less than, for example, 2.5 seconds, a small change in the design procedure yields the desired dynamic response.

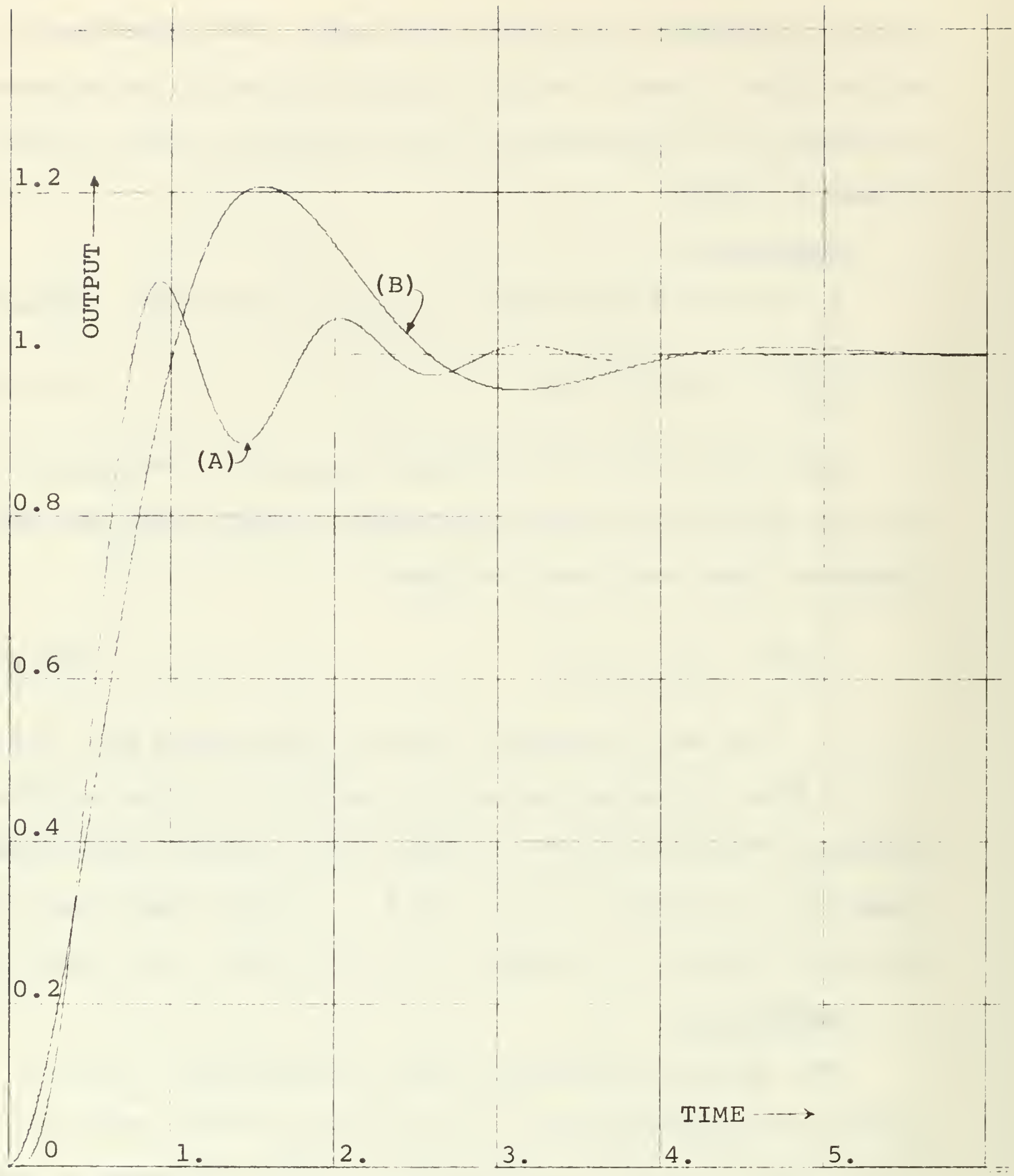


Fig. III.3. EXAMPLE 7. The fastest time response (A) and the desired response (B) (for EXAMPLE 6) to a unit step input. (SOLUTION 1).

For this application the performance index takes the form

$$J = x_7(t) = \int_0^{t_f} w_1(1.-x_1(t))^2 dt \;, \qquad \text{(III.29)}$$

where $t_f = 4$ seconds as before and w_1 is the penalty function defined as

$$w_1 = \begin{cases} 1 & \text{for} \quad t < 2.5 \\ 1000 & \text{for} \quad t \geq 2.5 \;. \end{cases} \qquad \text{(III.30)}$$

This penalty function assures a settling time less than or equal to 2.5 seconds. Minimization of Eq. (III.29) yields the optimum parameter values; $K_a^* = 0.047$, $K_t^* = 0.441$. The output of the compensated system with optimum parameter values is shown in Fig. III.4.

<u>Solution 3</u>

A somewhat different approach to the same design problem is considered. The differences from the previous solutions are given below.

The system gain is used as the third free parameter (which is constrained in the region $10^3 \leq K \leq 10^4$).

The performance index

$$J = x_7(t) = \int_0^{t_f} \left[w_1(1.-x_1(t))^2 - x_2^2(t) \right] dt \qquad \text{(III.31)}$$

is used for the design. In this approach the time derivative of the output is maximized to obtain the fastest

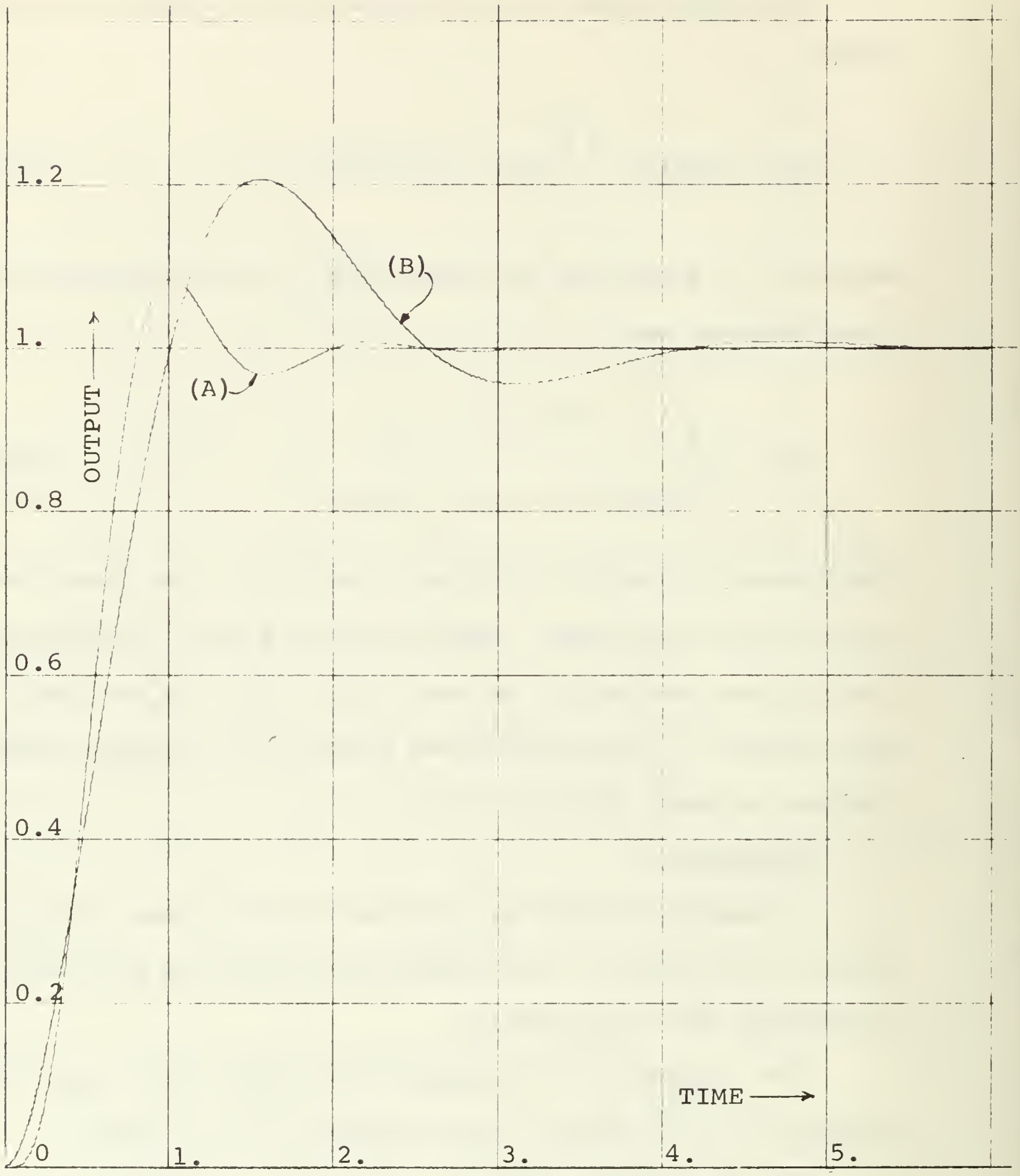


Fig. III.4. EXAMPLE 7. The fastest time response (A) (with a constraint on the settling time) and the desired response (B) (for EXAMPLE 6) to a unit step input. (SOLUTION 2).

response. The first term $(w_1(1.-x_1(t))^2)$ is used to satisfy another (assumed) performance specification which requires the peak overshoot be less than or equal to 40%; for this purpose the penalty function

$$w_1 = \begin{cases} 1 & \text{for } |x_1(t)| \le 1.4 \\ 100 & \text{for } |x_1(t)| > 1.4 \end{cases} \quad \text{for all } t \text{ in } [0, t_f] \tag{III.32}$$

is used.

Minimization of the performance index (Eq. (III.31)) yields the optimum parameter values $K^* = 1012.6$, $K_a^* = 0.07$, $K_t^* = 0.296$. The optimum response is shown in Fig. III.5.

H. EXAMPLE 8

In the Example 4 of Chapter II a third-order linear system was compensated for the given dominant root locations. The use of a lag compensator (in the s-domain) was demonstrated in Example 4. The same example is re-solved here to compare the results of the two methods.

The system is shown in Fig. II.11(a). Paragraph numbers are used to indicate the corresponding steps in Example 5.

1. The desired root locations in Example 4 ($\sigma_d = 1$, $\omega_d = \pm 2$) correspond to a second-order system which is defined by the state equations

$$\begin{aligned} \dot{x}_5(t) &= x_6(t) \\ \dot{x}_6(t) &= 5(r(t) - x_5(t)) - 2x_6(t) \ . \end{aligned} \tag{III.32}$$

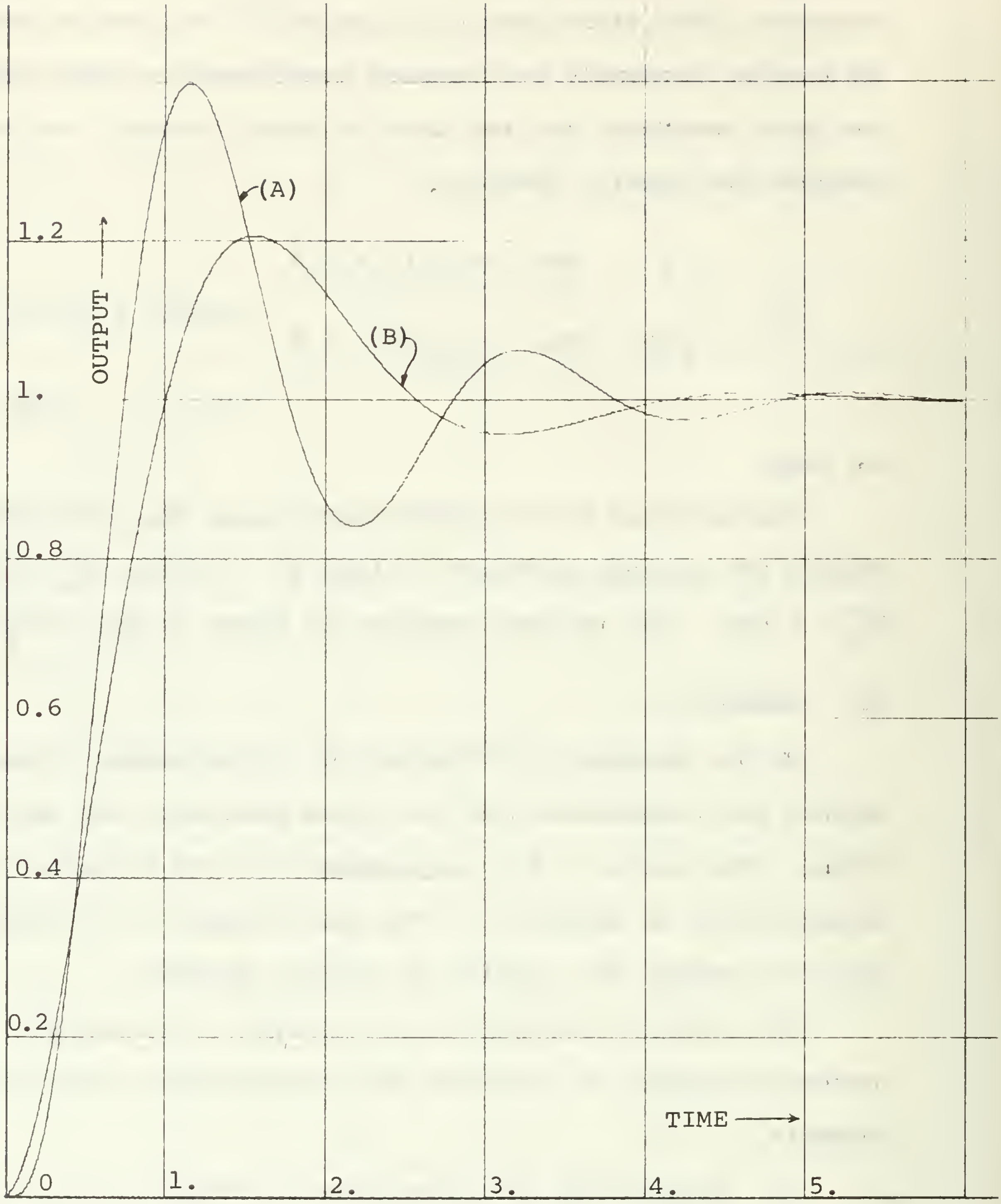


Fig. III.5. EXAMPLE 7. The fastest response (A) by maximizing the derivative term, and the desired response (B) (for EXAMPLE 6) (SOLUTION 3).

The model response is

$$c_m(t) = x_5(t) \ . \tag{III.33}$$

The unit step input is used in the design, i.e.

$$r(t) = \mathbb{1}(t) \ . \tag{III.34}$$

2. - 3. A cascade compensator is used (as shown in Fig. II.11(b)) to obtain the desired response. The state equations of the system are written from the system's transmission function (Eq. (II.45)) in the rational canonical form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{b}r(t) \tag{III.35}$$

where

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 \end{bmatrix} , \tag{III.36}$$

$$\underset{\sim}{b} = (0 \ 0 \ h_3 \ h_4)^T \text{ , and}$$

$$\begin{aligned} a_o &= Kp & a_3 &= 12 + p \\ a_1 &= 36p+K/k & h_3 &= K/k \\ a_2 &= 36+12p & h_4 &= a_o - h_3a_3 \ . \end{aligned} \tag{III.37}$$

The output of the system is

$$c_s(t) = x_1(t) \ .$$

4. The time derivative of the performance index is

$$\dot{x}_7(t) = (x_1(t) - x_5(t))^2 \ . \tag{III.38}$$

5. The constraints on the free system variables are

$$\begin{aligned} &425 \le K \le 1000 \\ &0.1 \le k \le 10 \\ &0<p \le 100 \text{ (arbitrary)} \ . \end{aligned} \tag{III.39}$$

6. The performance index defined by the integral of Eq.(III.38) is minimized with respect to the free system parameters and the optimum parameter values are $K^* = 554$, $k^* = 7.84$, and $p^* = 0.0116$. The optimum response to a unit step input and the model response are shown in Fig. III.6.

I. CONCLUDING REMARKS

In this chapter, the method "Optimization for the Best Response in the Time-Domain" has been developed and applied to several linear system design problems. The same examples, given in Chapter II for the s-domain design method, have been reconsidered here to make comparison of the two design methods possible. From this comparison and from the general development of the method, it may be concluded that this method has many advantages over the present dynamic system design procedures. The most important advantages are listed below:

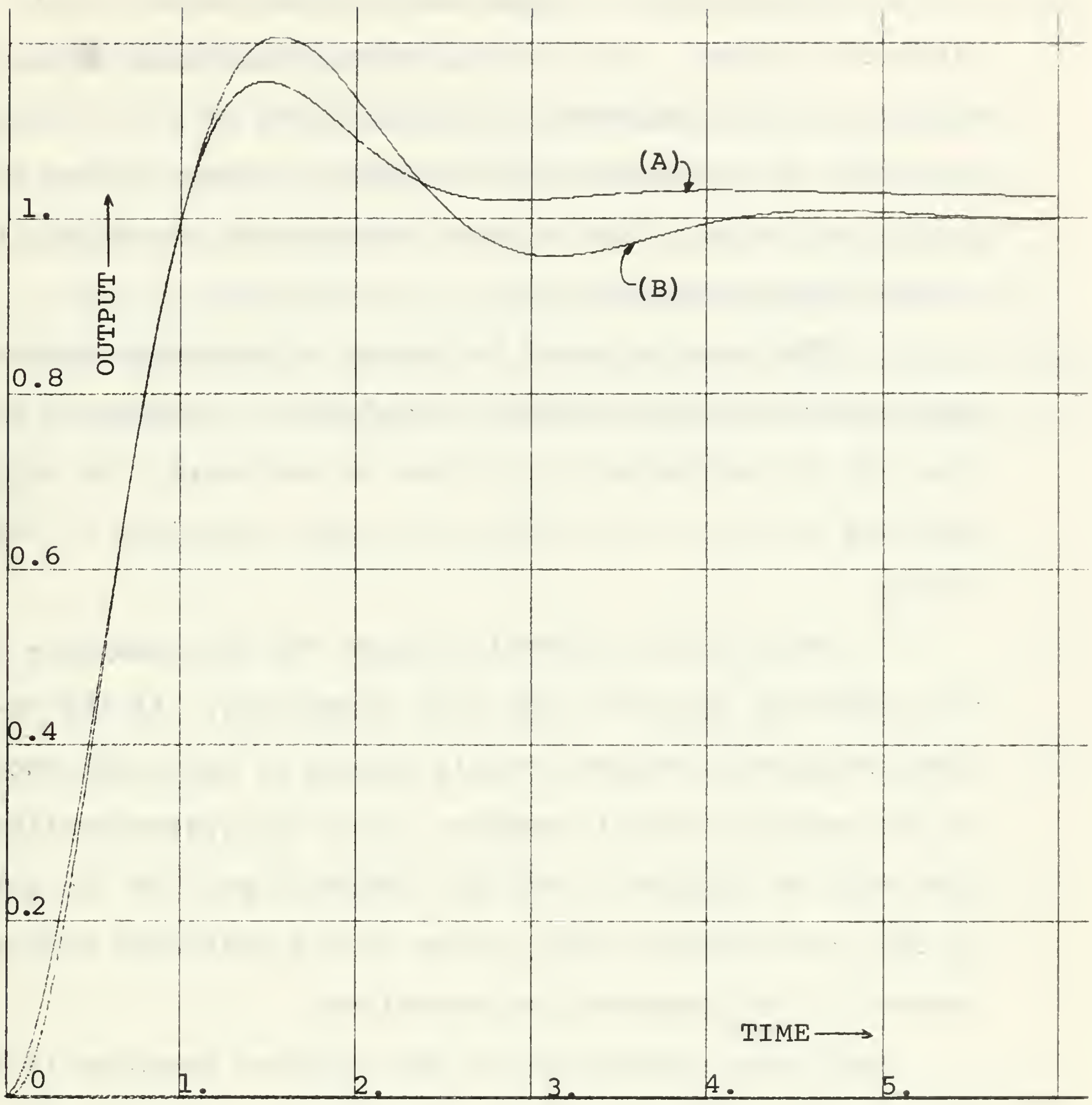


Fig. III.6. EXAMPLE 8. Time response (A) of the compensated system with a lag compensator and the desired response (B), to a unit step input.

1. This method is applicable to high-order, multi-parameter systems. Any dynamic system which can be represented by state equations can be designed by this method; the order of the system and the number of free system parameters can be very high without introducing any difficulty in the design procedure.

2. The work required to design a high-order system is less than with other methods; in fact, if a computer program for the minimization process is available, the only required work is in obtaining the state equations of the system.

3. This method directly shapes the time response of the system by adjusting the free parameters. If the optimum response obtained by this method is not close enough to the desired (model) response, then the system configuration must be changed. The only indirect part of the method is that the designer must decide on the available system components and compensation techniques.

The method "Optimization for the Best Response in the Time-Domain" enjoys the following advantages over the s-domain method developed in Chapter II.

a. The basic idea behind this method is much simpler.

b. Optimization for the best response in the time domain yields a better answer to the design problem because it is more direct and it does not employ any questionable assumptions such as dominancy of the desired roots.

c. This method requires less hand calculation and less programming effort.

d. This method is not limited to the design of linear systems. Besides its simplicity this is the most important advantage of the method, because it provides a powerful computerized approach to the design of nonlinear and sampled-data systems and also, for example, systems with time delay. These applications are presented in Chapter IV.

IV. OPTIMIZATION FOR THE BEST RESPONSE IN THE TIME DOMAIN -- APPLICATIONS TO NONLINEAR AND SAMPLED-DATA SYSTEMS AND SYSTEMS WITH TIME DELAY

A. GENERAL

The method "Optimization for the Best Response in the Time-Domain" is applicable to a large variety of dynamic system synthesis problems. Any dynamic system which can be represented by state equations can be designed to yield a desirable dynamic and steady-state performance.

In this chapter, applications of the method to non-linear and sampled data systems and to systems with time delay are considered. Three simple, but representative, examples are given. Application to these classes of systems increases the value of the method, because direct and efficient design methods are rare in these areas of control theory.

If the system is not linear, it may require some additional logic (in the computer programming) to obtain the state values in the time interval of interest; but the main procedure (as presented in Chapter III) is directly applicable.

B. EXAMPLE 9

A chemical process with an inherent time delay ($T = 0.5$ seconds) is to be regulated. The system is shown in Fig. IV.1. The load changes are sudden and can be assumed to be step functions. The only available free parameter is the amplifier gain (K_c). The desired dynamic and steady-state specifications will be defined later.

For the given system configuration the system's response can be divided into two parts:

1. Steady-State Accuracy. Since the given system is type-0, there will always be a constant steady-state error for step disturbances. The value of the steady-state error (E_{ss}), which depends on the amplifier gain (K_c), is

$$E_{ss}(K_c) = 1/(1+4.8\ K_c) \qquad \text{(IV.1)}$$

2. Dynamic Accuracy. This is also a function of the amplifier gain (K_c). The dependence of the shape of the dynamic response (on K_c) will be apparent at the end of the solution.

In general, high gain values cause more oscillations, and less steady-state error (see Eq. (IV.1)); low gain values result in a less oscillatory response but more steady-state error. With only one free parameter, it is impossible to meet all of the specifications; therefore, the design method should seek the best compromise between the desired specifications.

To make the initial conditions of the states all zero, it can be assumed that the reference signal is zero ($u(t)=0$), and that the system is in a steady-state condition. The system's behavior in case of sudden load changes will be investigated.

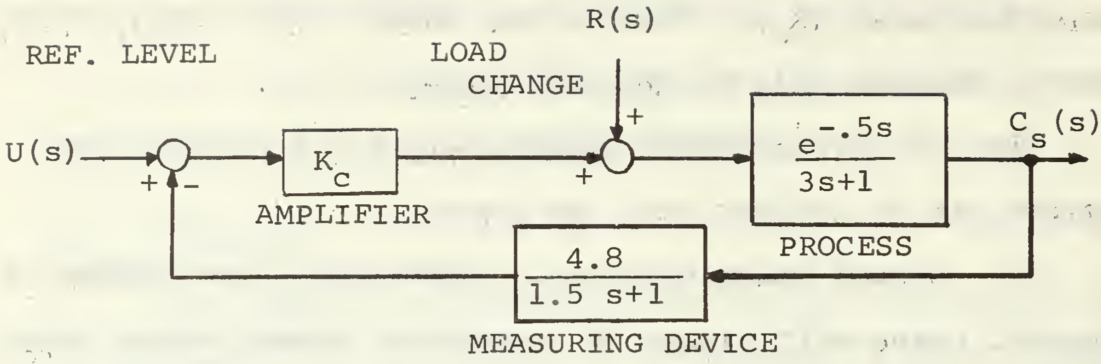


Fig. IV.1. EXAMPLE 9

The state equations of the system are

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + r(t-0.5)/3 \\ \dot{x}_2(t) &= -(9.6K_c\, x_1(t-0.5) + 2x_1(t) + 9x_2(t) + r(t-0.5))/9 \end{aligned} \tag{IV.2}$$

There are three terms affected by the 0.5 - second time delay. $x_1(t-0.5)$ represents the value of $x_1(t)$, 0.5 seconds earlier during the process. The load disturbances are assumed to be unit step functions; therefore,

$$r(t-0.5) = \begin{cases} 0 & \text{for} \quad t < 0.5 \\ 1 & \text{for} \quad t \geq 0.5 \end{cases} \tag{IV.3}$$

can be used.

The output of the system is

$$c_s(t) = x_1(t) \ . \tag{IV.4}$$

It the performance specifications only require that the output of the system should be kept as close to the reference as possible, and the transients should last not more than 30 seconds, then a performance index, which measures the deviation of the output from the reference level (from $t = 0$ to $t_f = 30$ seconds) can be defined as

$$J(K_c) \triangleq x_3(t) \triangleq \int_0^{t_f} (x_1(t) - u(t))^2 dt \quad . \qquad \text{(IV.5)}$$

Since the reference level was assumed to be zero, this performance index becomes

$$x_3(t) = \int_0^{t_f} x_1^2(t)\ dt \quad . \qquad \text{(IV.6)}$$

The time derivative of the performance index is

$$\dot{x}_3(t) = x_1^2(t) \quad . \qquad \text{(IV.7)}$$

Minimization of this performance index with respect to K_c yields the "best" response of the system within the limitations of the system configuration.

Eqs. (IV.2) and (IV.7) are integrated with respect to time to calculate the value of the performance index at each step of minimization process. The integration step size is chosen as an integer submultiple of the time in the process ($\Delta t = 0.05$ is chosen). For the first ten integration steps the values of $r(k\,\Delta t - 0.5)$ and $x_1(k\,\Delta t - 0.5)$

are zero[1]; the value of the variable $x_1(k\,\Delta t)$ are stored (k is the integrating step counter). After the tenth integration step $(k > 10)$, $r(k\,\Delta t - 0.5)$ is equated to unity and the values of $x_1((k-10)\Delta t)$ are found and used from the stored values of $x_1(k\,\Delta t)$.

The only difference from a standard linear system design problem is in the logic of expressing the existing time delay in the system, and as explained above, this does not introduce any complication to the solution.

$K_c^* = 1.66$ is found by the minimization procedure. The time response of the system to a unit step load change is shown in Fig. IV.2. Peak overshoot is about 33% and the steady-state error is 11.1%.

If the desired response is to be less oscillatory, one can include the time derivative of the output in the performance index; that is,

$$\dot{x}_3(t) = x_1^{\,2}(t) + \dot{x}_1^{\,2}(t) \,, \qquad \text{(IV.8)}$$

or by substituting the value of $\dot{x}_1(t)$ (from Eq. (IV.2)) one obtains

$$\dot{x}_3(t) = x_1^{\,2}(t) + (x_2^{\,2}(t) + r(t - 0.5)/3)^2 \,. \qquad \text{(IV.9)}$$

---

[1] Since the numerical integration is a discrete process, the arguments of the delayed variables, $r(t-0.5)$ and $x_1(t-0.5)$ are represented in discrete form.

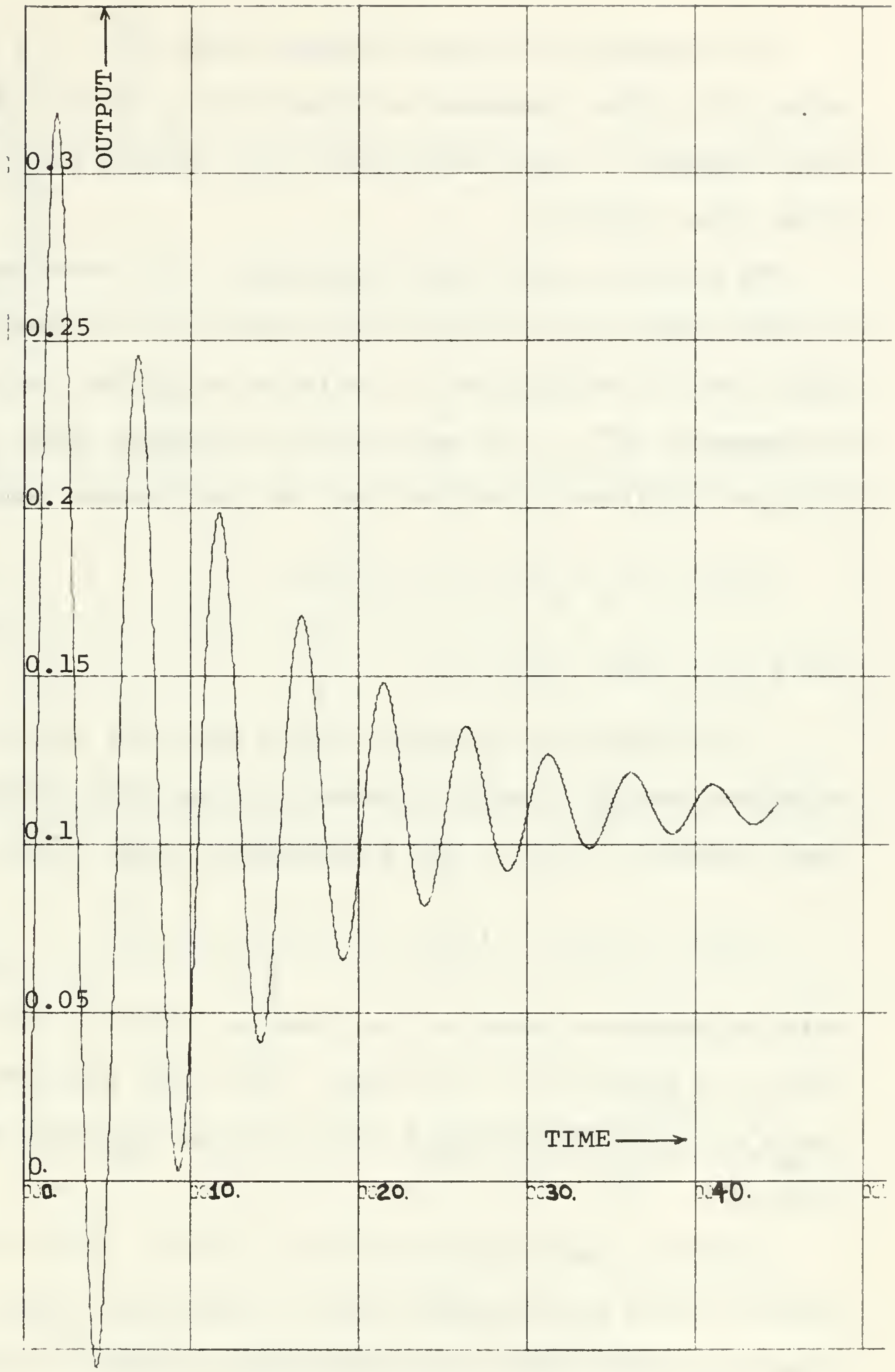


Fig. IV.2. EXAMPLE 9. Time response of the system to a unit step disturbance ($\dot{J} = x_1^2$).

By minimizing this performance index $K_c^* = 1.5$ is obtained (The time response is shown in Fig. IV.3). This time response is less oscillatory but suffers more steady-state error (12.2%).

To meet the given specifications, the terms representing the output and its time derivative (in the performance index) may be multiplied by suitable weighting factors. For example, $K_c^* = 1.12$ and the time response shown in Fig. IV.4 are obtained by minimizing the performance index

$$\dot{x}_3(t) = w_1 \, x_1^{\,2}(t) + w_2 \, \dot{x}_1^{\,2}(t) \, , \tag{IV.10}$$

for $w_1 = 1$ and $w_2 = 10$.

If the desired response should have the smallest cumulative rate of change (slowest varying transients in the time interval $[0, t_f]$), the performance index takes the form

$$\dot{x}_3(t) = \dot{x}_1(t) = (x_2(t) + r(t - 0.5)/3)^2 \; . \tag{IV.11}$$

This performance index is the same as (IV.10), of course, when $w_1 = 0$ and $w_2 = 1$ is used. With this performance index $K_c^* = 0.473$ is obtained (for the time response see Fig. IV.5).

Finally some applications may require that the system's output should not deviate from the reference level more than a certain percentage for a given load change. To satisfy this requirement a constraint on the output is inserted.

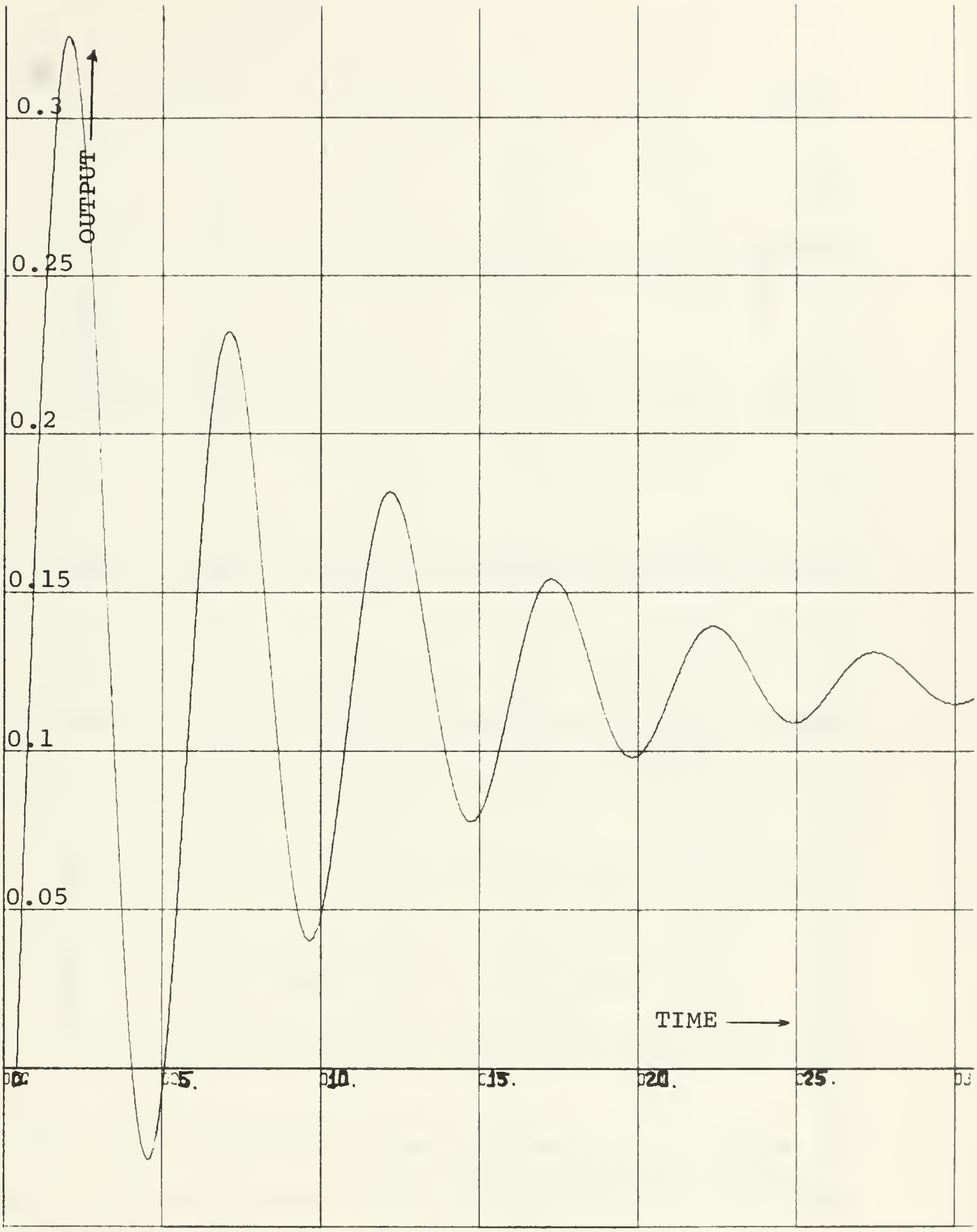


Fig. IV.3. EXAMPLE 9. Time response of the system to a unit step disturbance ($\dot{J} = x_1^{\,2} + \dot{x}_1^{\,2}$).

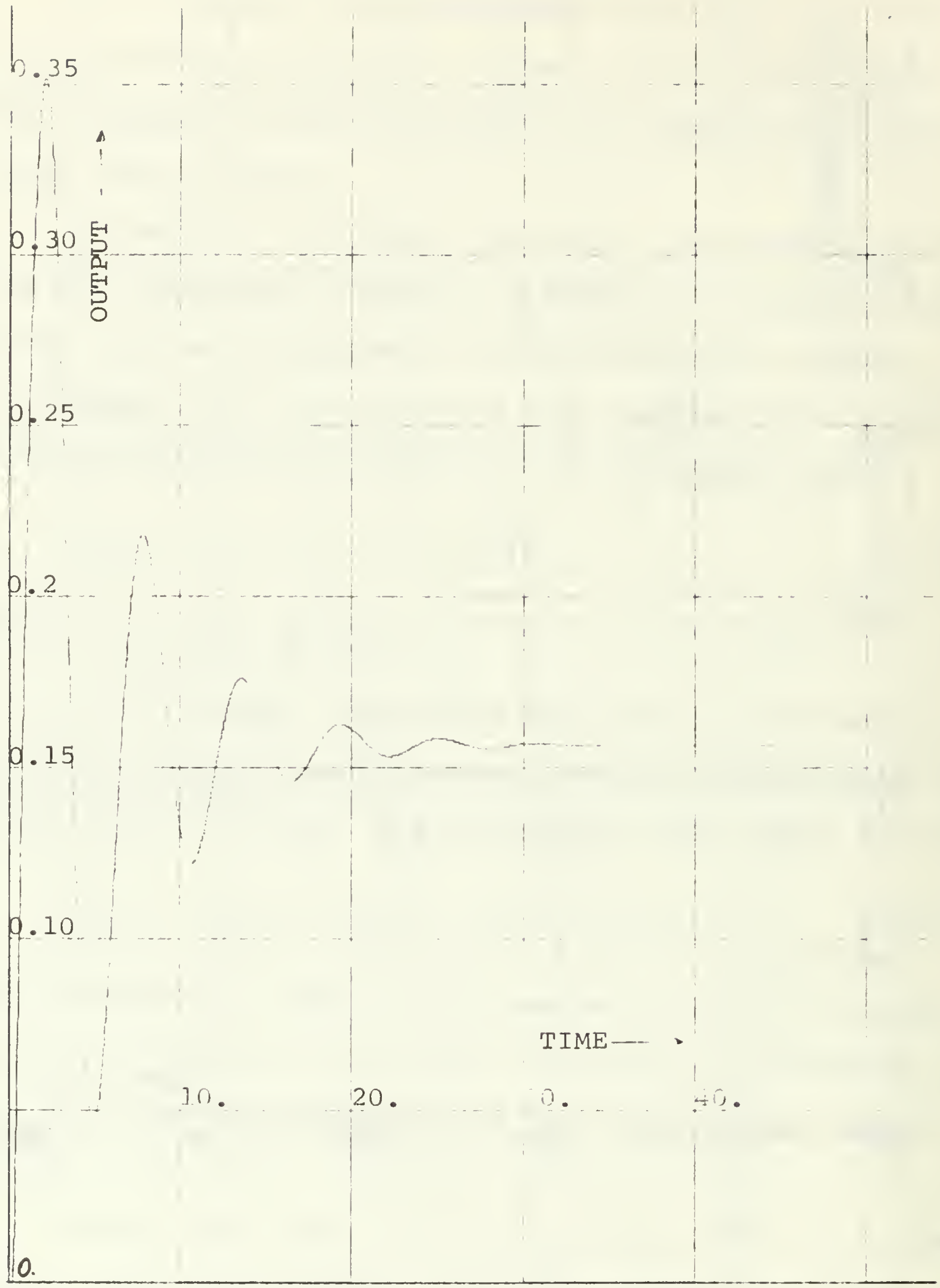


Fig. IV.4. EXAMPLE 9. Time response of the system to a unit step disturbance ($\dot{J} = x_1^2 + 10\dot{x}_1^2$).

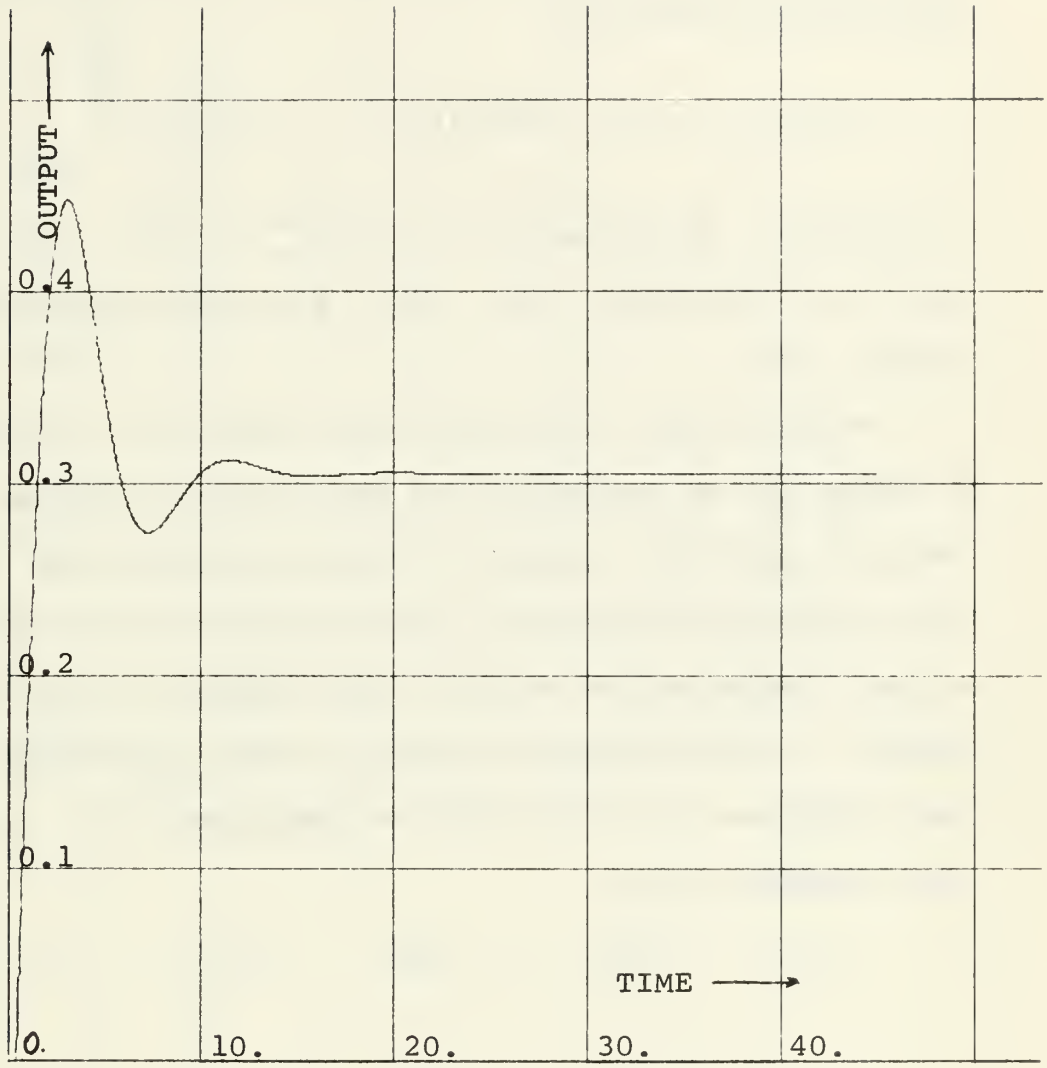


Fig. IV.5. EXAMPLE 9. Time response of the system to a unit step disturbance ($\dot{J}=\dot{x}_1^2$).

For example, to ensure that the maximum deviation from the reference level is less than 31% of the load disturbance, the performance index defined in Eq. (IV.8) is used and the additional constraint

$$|x_1(t)| \leq 0.31 \quad \text{for all} \quad t \quad \text{in } [0, t_f] \qquad \text{(IV.12)}$$

is imposed. Minimization[1] yields $K_c^* = 1.85$; the time response of the system (for a unit step load change) is shown in Fig. IV.6.

This series of solutions (see Table IV.1) shows that a system can be optimized for any desired response (that is, the most suitable response within the limitations of the given system configuration can be obtained) by using the method "Optimization for the Best Response in the Time-Domain" -- the existence of one or more time delays does not introduce any difficulty or complication in the solution process.

---

[1] The penalty function technique used in Example 7 was applied.

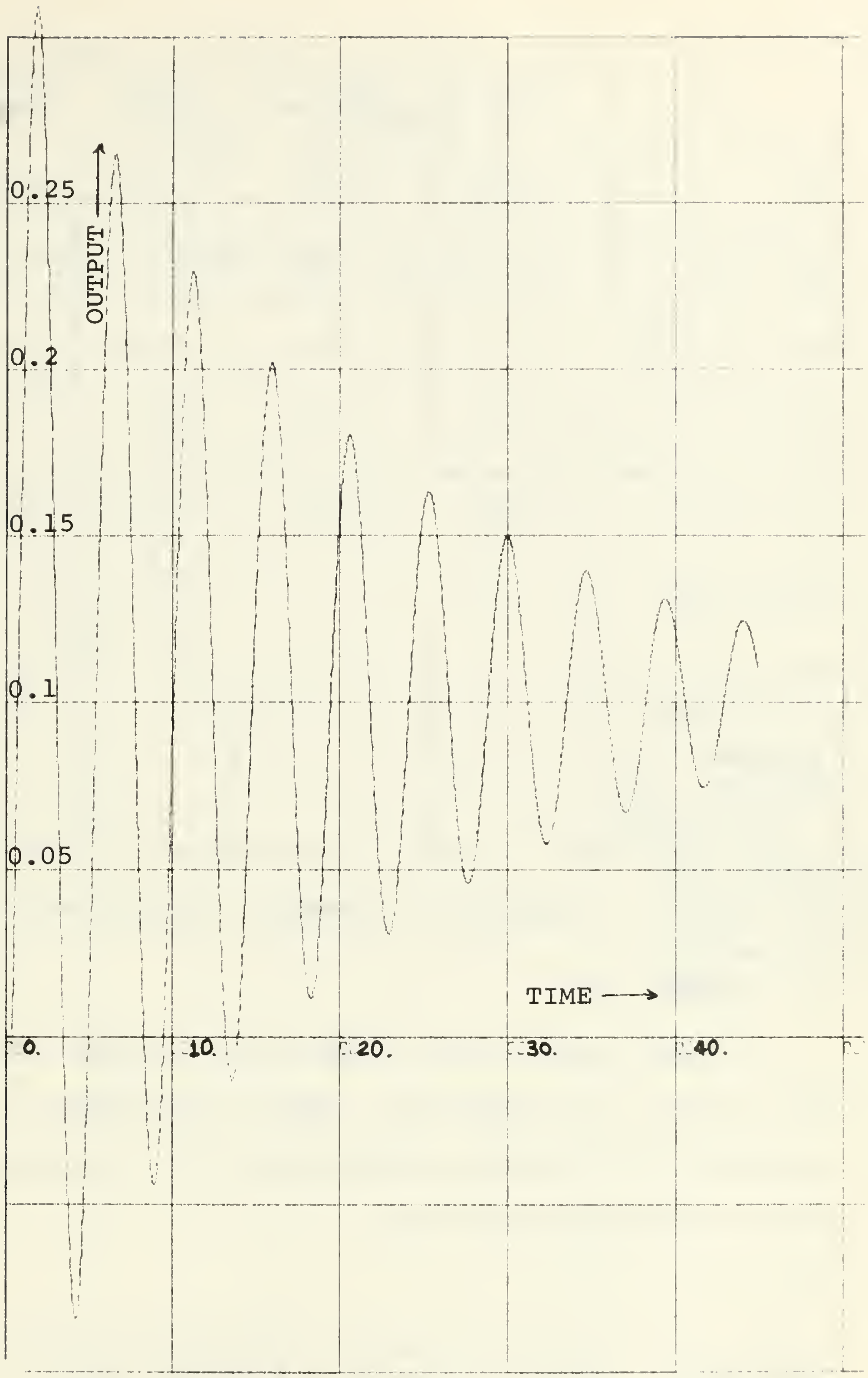


Fig. IV.6. EXAMPLE 9. Time response of the system to a unit step disturbance ($\dot{J} = x_1^2 + \dot{x}_1^2$), ($|x_1| \leq 31\%$).

J	K_c^*	Peak overshoot	E_{ss}	Time response
x_1^2	1.66	33%	11.1%	Fig. IV.2
$x_1^2 + \dot{x}_1^2$	1.5	35%	12.2%	Fig. IV.3
$x_1^2 + 10\,\dot{x}_1^2$	1.12	36%	15.6%	Fig. IV.4
$\dot{x}_1^2$	0.473	45%	30%	Fig. IV.5
$x_1^2 + \dot{x}_1^2$ $\lvert x_1 \rvert \le 31\%$ for all t in the time interval $[0, t_f]$	1.85	31%	10.1%	Fig. IV.6

TABLE IV.1. EXAMPLE 9 (SOLUTIONS)

C. EXAMPLE 10

A simple second-order sampled-data system (shown in Fig. IV.7(a)) is considered. The two adjustable design parameters (free system parameters) are the system gain (K) and the sampling period[1] (T).

---

1 The method "Optimization for the Best Response in the Time-Domain" yields an important advantage when it is applied to the sampled-data systems -- the sampling period can be taken as one of the free variables without increasing the complexity of the solution.

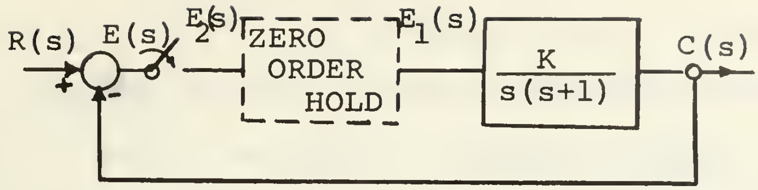


Fig. IV.7(a). EXAMPLE 10. The system.

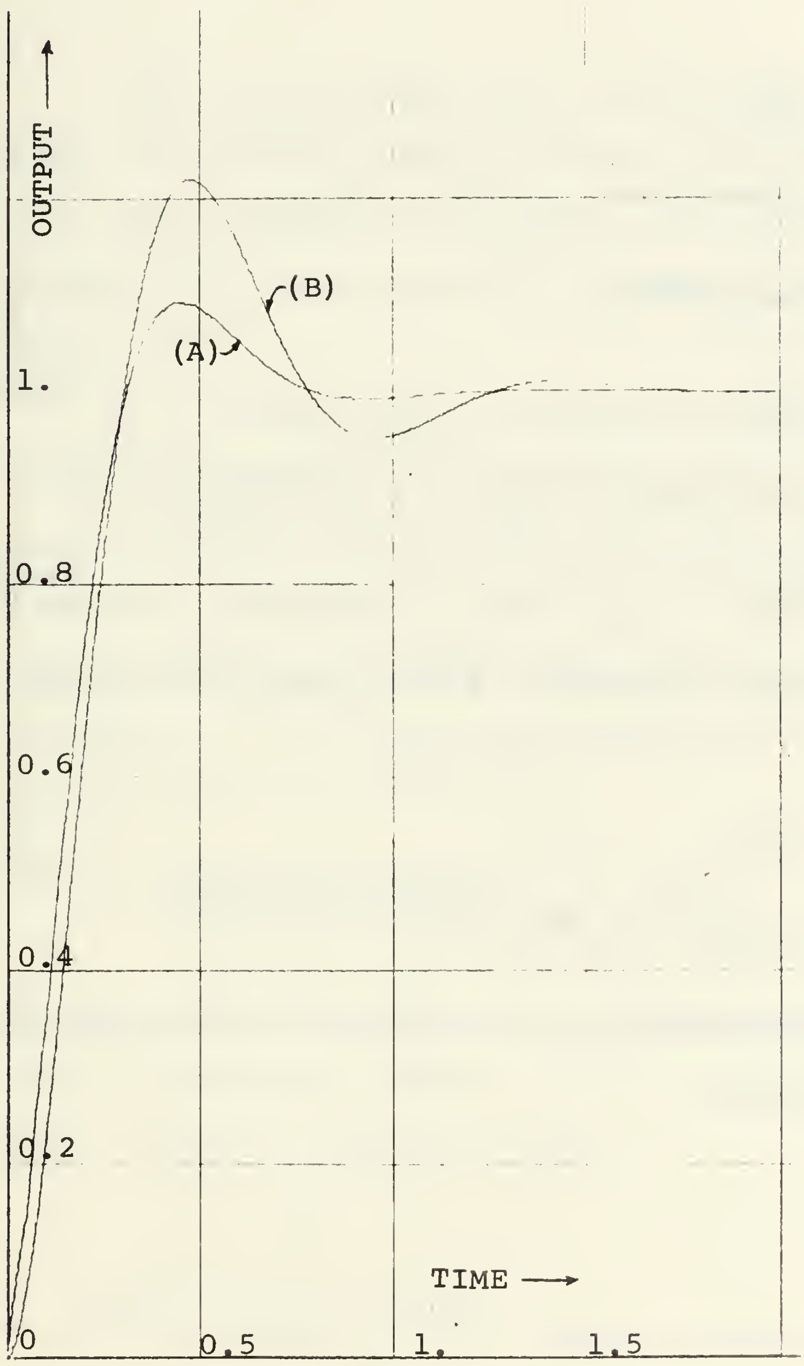


Fig. IV.7(b).
EXAMPLE 10.
Time response of the system (A) without a hold circuit and the desired response (B).

Solution (without a Hold Circuit)

1. The dynamic performance specifications[2] require that the system's response should be as close to the response of the second-order continuous system as possible. The state equations of the second-order continuous system are

$$\dot{x}_3(t) = x_4(t)$$
$$\dot{x}_4(t) = 0.526(r(t) - x_3(t)) - 0.630x_4(t) \ , \qquad \text{(IV.13)}$$

and the model response is

$$c_m(t) = x_3(t) \ . \qquad \text{(IV.14)}$$

The unit step input is used in the design, i.e.

$$r(t) = \mathbb{1}(t) \ . \qquad \text{(IV.15)}$$

3. The state equations of the system are written from Fig. IV.7(a)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = K\, e_1(t) - x_2(t) \ . \qquad \text{(IV.16)}$$

Since the hold circuit is not used in this solution (see Fig. IV.7(a)).

---

[1] Paragraph numbers indicate the corresponding steps in Example 5.

$$e_1(t) = e_2(t) \,, \tag{IV.17}$$

and also

$$e_1(t) = \begin{cases} 0 & \text{if the sampling switch is open} \\ r(t)-x_1(t) & \text{during the sampling instant} \end{cases} \tag{IV.18}$$

Logic must be included, in the computer program, to find the sampling instants, during the integration; in fact this is the only required addition to the computer program which is written for a standard linear system design.

It is assumed that the sampling switch is closed for the first time at $t = 0$, and opened after an integration step. In other words at the first step of the discrete numerical integration process $e_1(t) = r(t) - x_1(t)$ is used, then $r_1(t)$ is set to zero. Numerical integration is continued for "I - 1" steps with the $e_1(t) = 0$ value. I is an integer found from the equation

$$I = \frac{\text{Sampling Period}}{\text{Integration step size}} = T/\Delta t \tag{IV.19}$$

A step counter (i) counts the integration steps starting from the sampling instant. $i = I - 1$ indicates another sampling instant; when this occurs

$$\left.\begin{aligned} & i = 0 \text{ , and} \\ & e_1(t) = e(t) - x_1(t) \end{aligned}\right. \tag{IV.20}$$

are set by the computer program, and a new count is started. The sampling time is assumed to be one integration step size in this solution. If the sampling time (the period of time that the sampling switch stays closed) should be long compared to the integration step size then another counter may be added to indicate the opening instants of the sampling switch. The integration step size should be chosen as an integer submultiple of both the sampling period and sampling time.

The output of the system is

$$c_s(t) = x_1(t) \ . \tag{IV.21}$$

4. The time derivative of the performance index is

$$\dot{x}_5(t) = (x_1(t) - x_3(t))^2 \ . \tag{IV.22}$$

The final time of the integration is taken as the settling time of the continuous model,

$$t_f = 4/\sigma_d = 12.7 \text{ seconds} \ . \tag{IV.23}$$

5. Constraints on the free variables are

$$\begin{aligned} &0.1 \le T \le 1.5 \\ &0 < K \le 15 \ . \end{aligned} \tag{IV.24}$$

6. The performance index defined by the integral of Eq. (IV.22) is minimized with respect to the free system parameters (T and K) and the optimum parameter values are

$T^* = 0.6$, $K^* = 7.20$. The output of the system to a unit step input and the model response are shown in Fig. IV.7(b).

Solution (With a Hold Circuit)

If a zero-order hold circuit is inserted after the sampling switch the only necessary change is in the logic given in conjunction with the Eq. (IV.18).

After a sampling instant, $e_2(t)$ is not equated to zero, but it is retained at the value of the last sampling for the remaining I - 1 steps of integration. In this solution impulse sampling is used, i.e., the switch is not kept closed for 0.05 seconds but it is opened as soon as the sample is measured.

Minimization of the same performance index (Eq.(IV.22)) yields $T^* = 1.05$ and $K^* = 0.5875$. The output of the sampled data system with zero-order hold and the model response are shown in Fig. IV.8.

D. EXAMPLE 11

A designer simulates the fourth-order linear system shown in Fig. IV.9 and obtains the time response to a unit step input shown in Fig. IV.10. This response is for a system gain of K = 1600. It is assumed that this dynamic response perfectly fits the particular application, except that the velocity coefficient must be doubled for steady-state accuracy. When the gain is doubled (K = 3200) however, the system moves to its stability limit, and the time response is quite oscillatory as shown in Fig. IV.10.

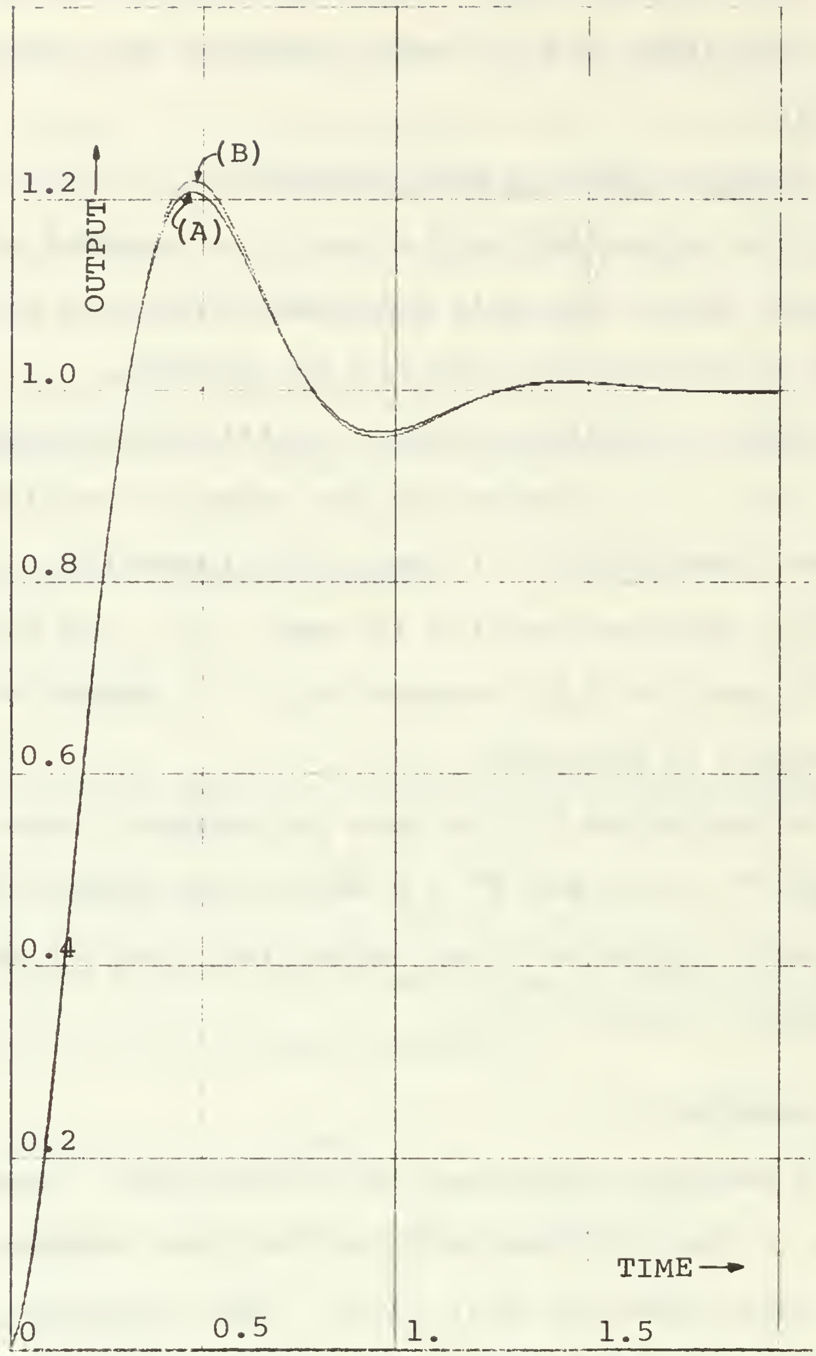


Fig. IV.8. EXAMPLE 10. Time response of the system (A) with a zero order hold circuit, and the desired response (B).

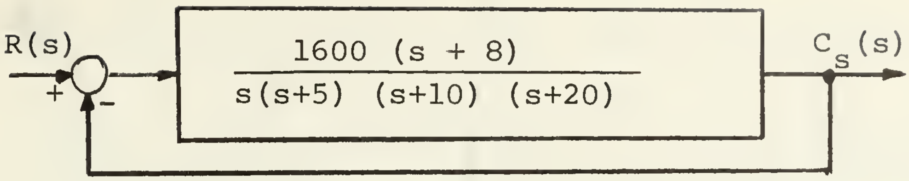


Fig. IV.9. EXAMPLE 11. The linear system.

The real system has two inherent nonlinearities -- a saturating amplifier in the forward path and dead zone in the feedback path. The real system with the nonlinearities is shown in Fig. IV.11. The designer decides to use two stages of linear cascade compensators to obtain a time response as close to the response of the linear system as possible; in other words, the linear system shown in Fig. IV.9 (with K = 1600) will be used to generate "the model response" for the optimization process.

The compensated system has four free parameters (see Fig. IV.12) -- k_1, k_2 (the compensation ratios of the first and second compensators) and p_1, p_2 (the pole locations of the compensators).

The state equations of the ideal (linear) system are

$$\begin{aligned}
\dot{x}_7(t) &= x_8(t) \\
\dot{x}_8(t) &= x_9(t) \\
\dot{x}_9(t) &= x_{10}(t) + K_i\,(r(t) - x_7(t)) \\
\dot{x}_{10}(t) &= -(1000x_8(t)+350x_9(t)+35x_{10}(t)+27K_i(r(t)-x_7(t))
\end{aligned} \tag{IV.25}$$

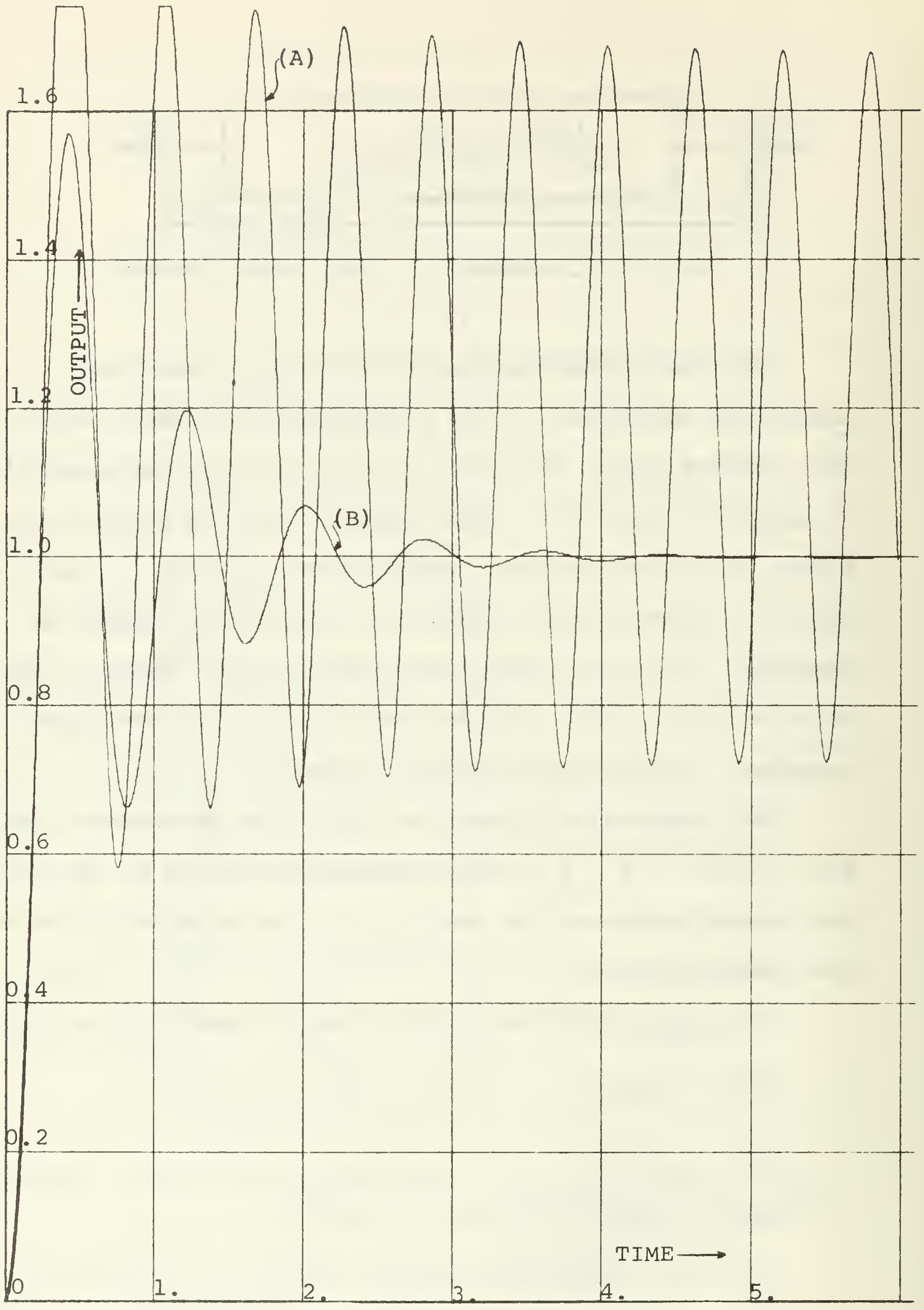
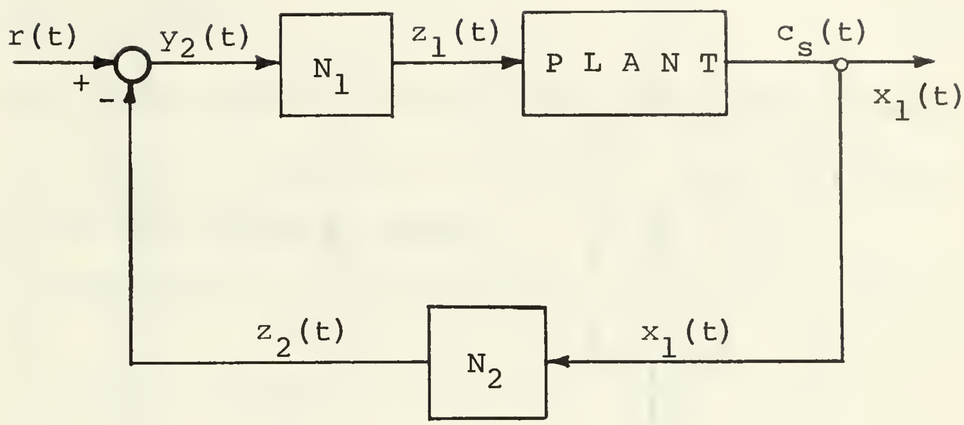


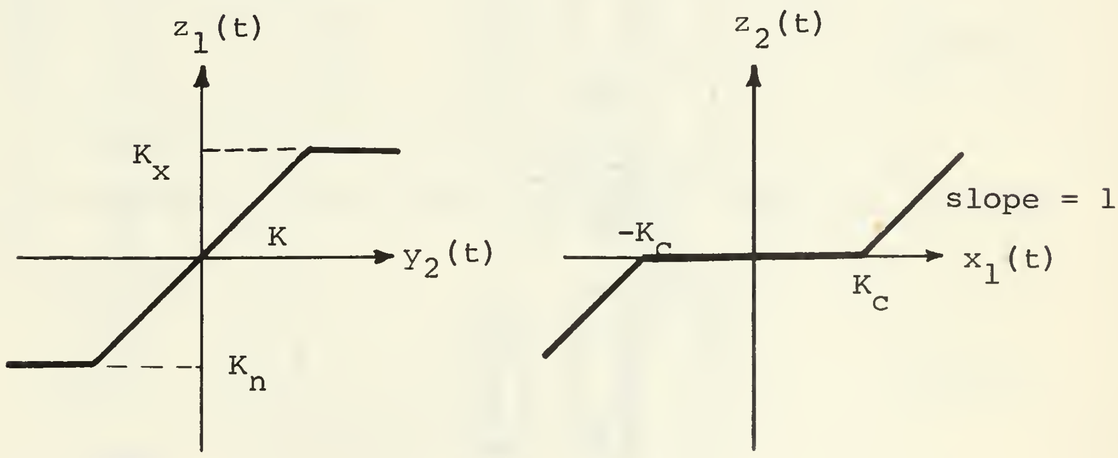
Fig.IV.10. EXAMPLE 11. Time response of the nonlinear system (A) (for K = 3200), and the time response of the linear system (B) (for K = 1600), (unit step input is used).



where the PLANT is defined by the transfert function

$$\frac{C_s(s)}{Z_1(s)} = G(s) = \frac{s+8}{s(s+5)(s+10)(s+20)}$$

(a) Real (Nonlinear) system



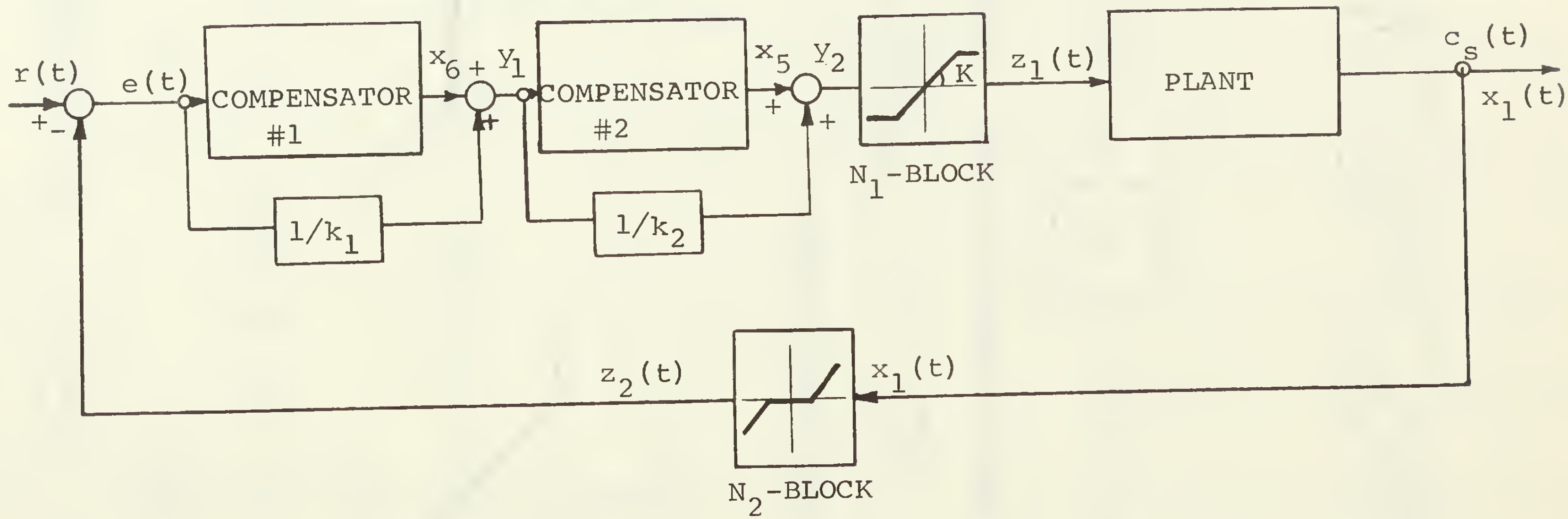
$K = 3200$, $K_x = 1600$, $K_n = -1400$

$K_c = 0.2$

(b) N_1 - Block

(c) N_2 - Block

Fig. IV.11. EXAMPLE 11. The real system.



where the PLANT is as defined in Fig. IV.11(a) and the compensator blocks are defined by the transfer functions

$$G_{c_i}(s) = \frac{p_i(1 - 1/k_i)}{s+p_i} \quad , \quad i = 1,2$$

Fig. IV.12. EXAMPLE 11. Compensated nonlinear system.

where $K_i = 1600$ and $r(t) = \mathbb{1}(t)$ -- the unit step function. The model response is the output of the linear system

$$c_m(t) = x_7(t) \quad . \tag{IV.26}$$

The state equations of the real (nonlinear) system with two cascade compensators (refer to Fig. IV.12) are written in the following steps:

1. The plant is specified by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= x_4(t) + z_1(t) \\ \dot{x}_4(t) &= -(1000x_2(t)+350x_3(t)+35x_4(t)+27\ z_1(t)), \end{aligned} \tag{IV.27}$$

and the output of the system is

$$c_s(t) = x_1(t) \quad . \tag{IV.28}$$

2. The Feedback Loop (N_2 - Block)

The output of the dead zone block (see Fig. IV.11(c)) can be represented as

$$z_2(t) = \begin{cases} 0 & \text{if } \ |x_1(t)| \le K_c \\ x_1(t)-K_c & \text{if } \ x_1(t) > K_c \\ x_1(t)+K_c & \text{if } \ x_1(t) < -K_c \ . \end{cases} \tag{IV.29}$$

3. <u>Error</u>

The error is given by

$$e(t) = r(t) - z_2(t) \quad . \tag{IV.30}$$

4. <u>Compensators</u>

The cascade compensators have transfer functions of the form

$$G_{c_i}(s) = \frac{1}{k_i} \frac{s + k_i p_i}{s + p_i} \quad , \quad i = 1,2 \tag{IV.31}$$

in the time domain; they can be represented by the configuration shown in Fig. IV.12. For the first compensator

$$\dot{x}_6(t) = -p_1 \, x_6(t) + p_1(1-1/k_1) \, e(t) \; , \tag{IV.32}$$

and its output is

$$y_1(t) = e(t)/k_1 + x_6(t) \; . \tag{IV.33}$$

Similarly, for the second compensator

$$\dot{x}_5(t) = - \, p_2(x_5(t) - (1-1/k_2) \, y_1(t)) \tag{IV.34}$$

and its output is

$$y_2(t) = y_1(t)/k_2 + x_5(t) \; . \tag{IV.35}$$

5. <u>N_1 - Block</u>

The output of the saturating amplifier is a nonlinear function of the magnitude of its input (see Fig. IV.11(b)); its output can be represented as

$$z_1(t) = \begin{cases} K_x & \text{if } \; z_1(t) \geq K_x \\ K_n & \text{if } \; z_1(t) \leq K_n \\ K \, y_2(t) & \text{otherwise} \end{cases} \tag{IV.36}$$

A performance index, which measures the difference of the outputs of the nonlinear and ideal systems, for a given magnitude of a step input (R), is defined as

$$J(R,\underset{\sim}{\alpha}) = x_{11}(t) = \int_{o}^{t_f} (x_1(t) - x_7(t))^2 dt \,, \qquad \text{(IV.37)}$$

and minimized with respect to the free system parameters ($\underset{\sim}{\alpha} = (k_1\ k_2\ p_1\ p_2)^T$) and the optimum parameter values ($\underset{\sim}{\alpha}^*$) are found.

Since the real system contains nonlinearities which are amplitude dependent, the performance index and the optimum parameter values are functions of the magnitude of the step input (R). For various amplitudes of step inputs the optimum parameter values are tabulated in Table IV.2. The time responses of the ideal and real systems for the conditions given in the last three lines of Table IV.2 are shown in Figs. IV.13, IV.14 and IV.15.

If it is known that the actual system is going to be operated usually with step inputs of a certain amplitude (say, R = 2.5), then the design can be completed by using the last line of Table IV.2; in this case the dynamic response of the system may not be satisfactory for step inputs much larger or smaller than the selected value (R = 2.5). To study the effect of the input magnitude, the pole locations are fixed at the optimum value found for the step input with 2.5 magnitude,

$$p = 10.2764 = \text{Constant} \,, \qquad \text{(IV.38)}$$

STEP INPUT MAGNITUDE (R)	k_1^* , k_2^*	p_1^* , p_2^*
0.5	1.7234	8.2500
1.0	1.2012	8.7051
2.0	1.1340	10.3790
2.5	1.1357	10.2764

TABLE IV.2

STEP INPUT MAGNITUDE (R)	k_1^* , k_2^*
0.1	1.8115
0.2	1.8008
0.3	1.7754
0.4	1.7480
0.5	1.7215
0.6	1.6904
0.7	1.2139
0.8	1.2100
0.9	1.2021
1.0	1.1885
1.5	1.1502
2.0	1.1336
2.5	1.1355
3.0	1.1512
3.5	1.1678
4.0	1.3162
4.5	1.3543
5.0	1.3787

TABLE IV.3

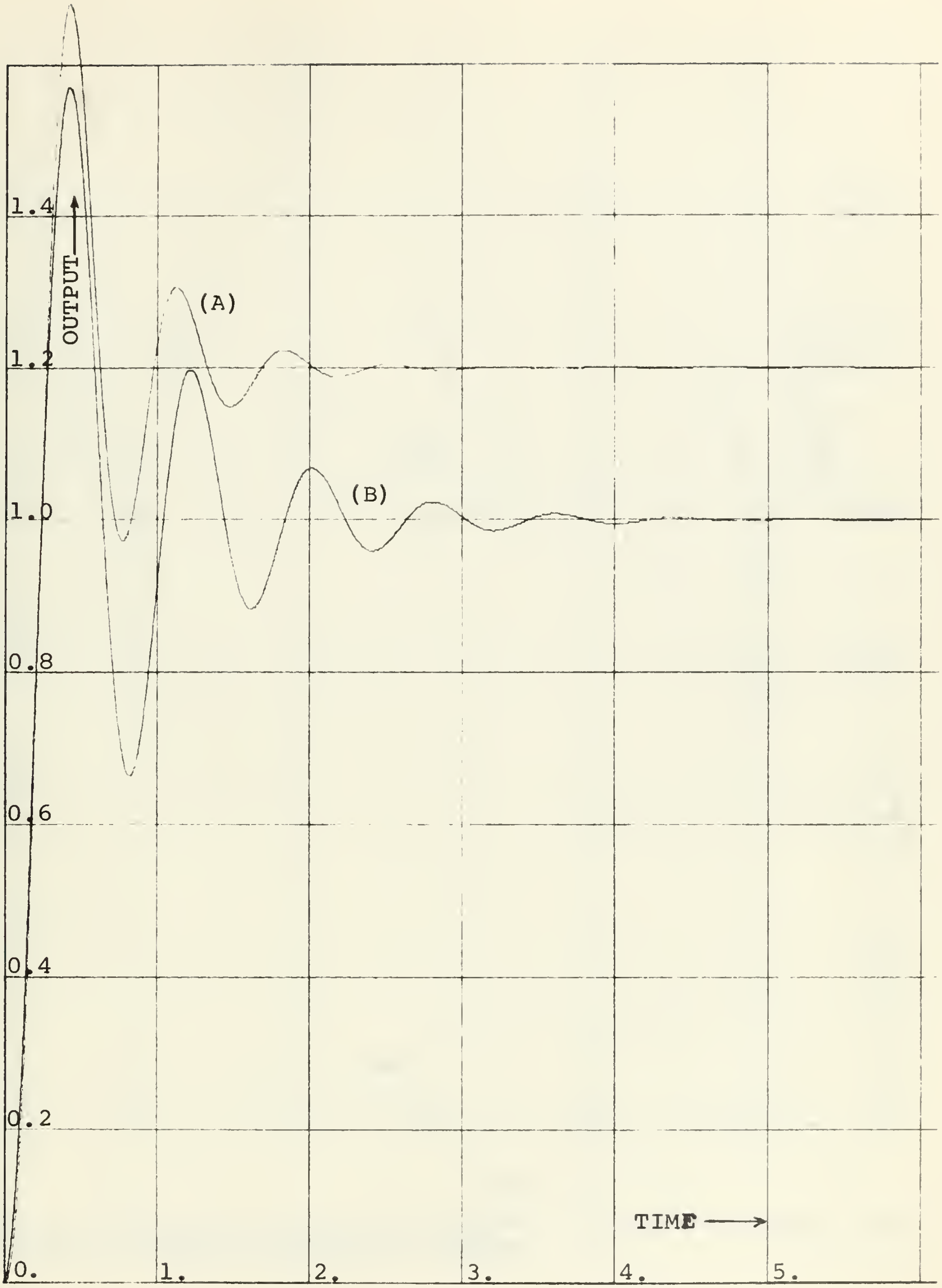


Fig. IV.13. EXAMPLE 11. Compensated (A) and desired (B) systems' responses ($r(t) = \mathbb{1}(t)$).

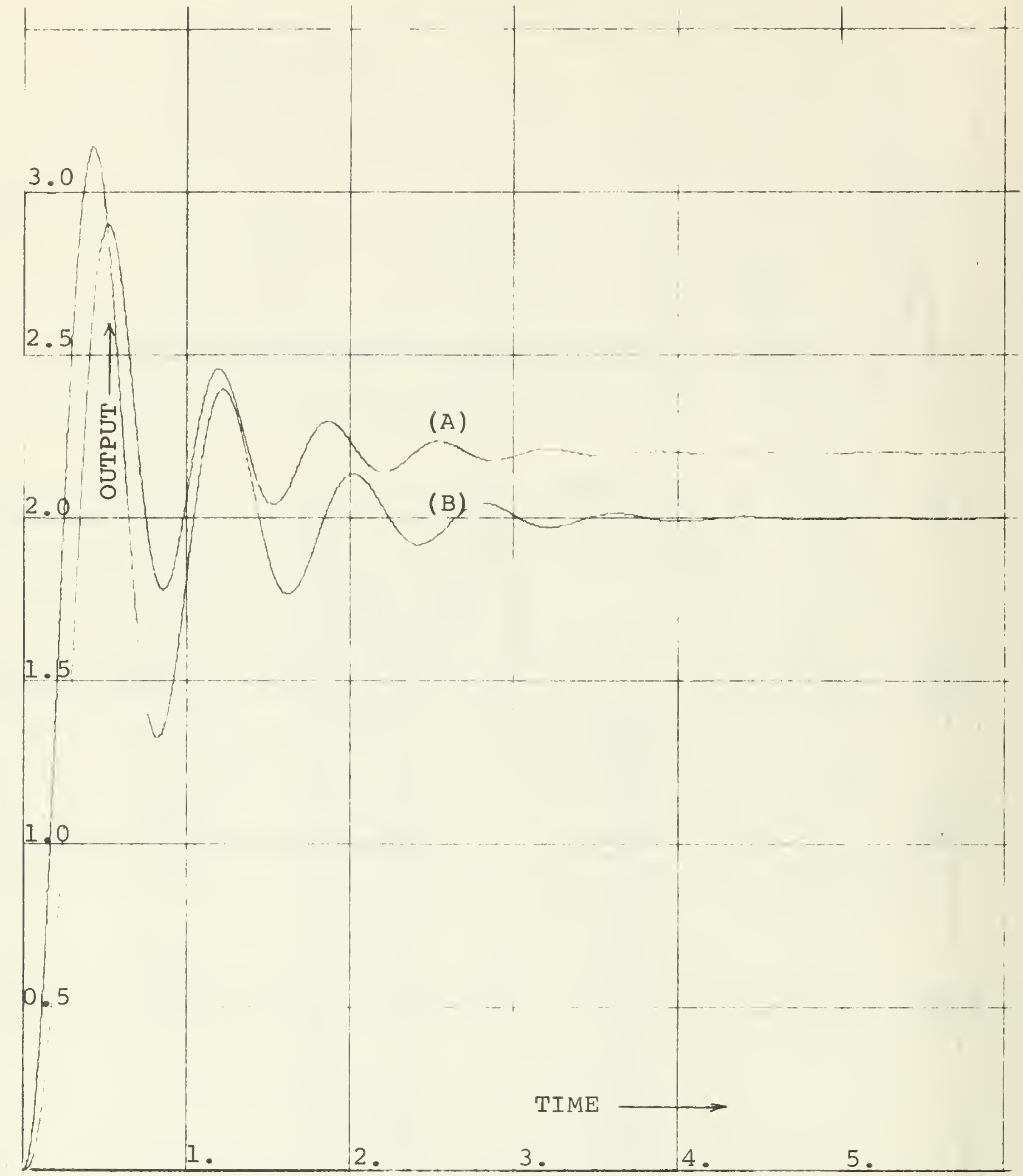


Fig. IV.14. EXAMPLE 11. Compensated (A) and the desired (B) systems' responses ($r(t) = 2 \times 1(t)$).

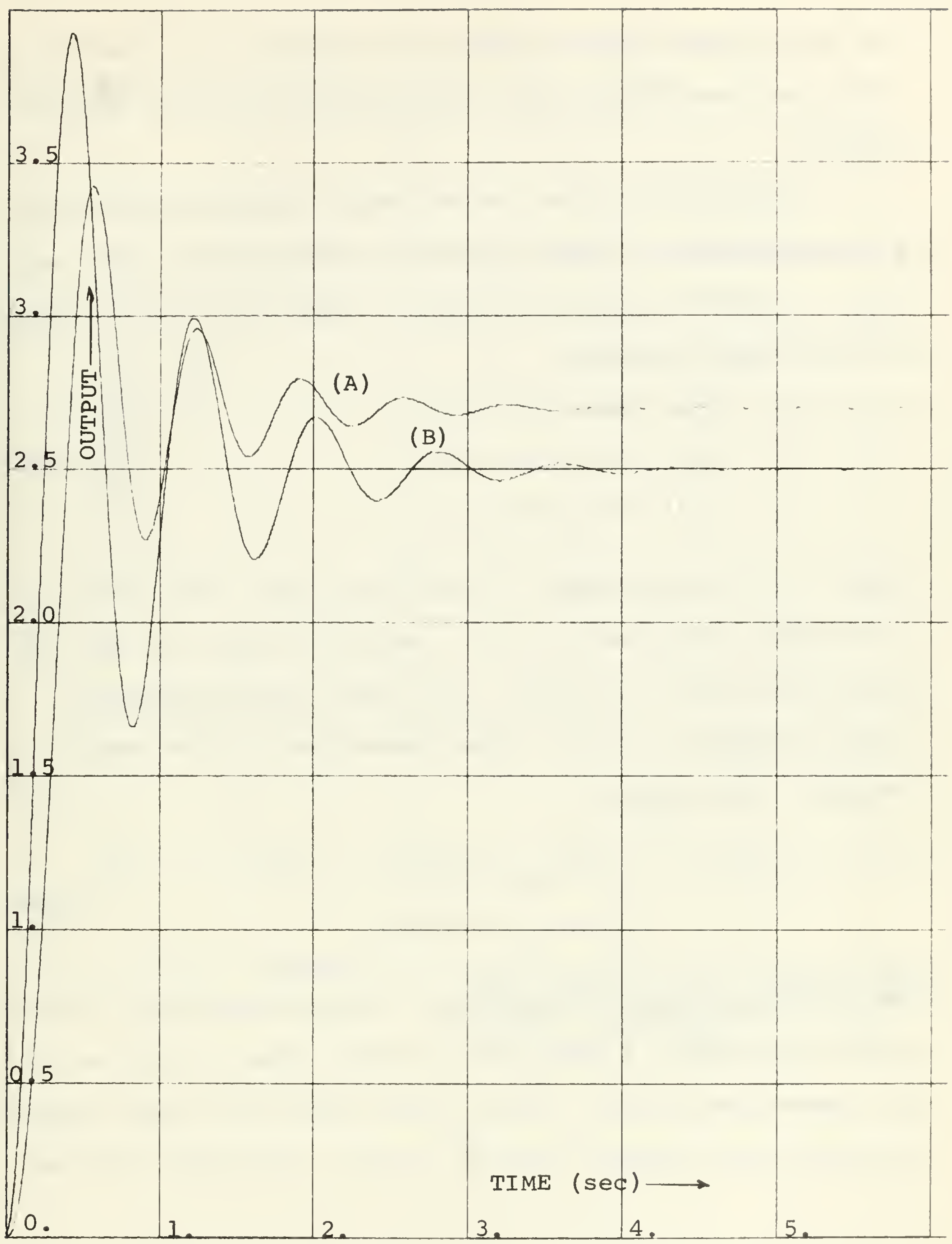


Fig. IV.15. EXAMPLE 11. Compensated (A) and the desired (B) systems' responses ($r(t) = 2.5 \times 1\!\!1(t)$).

and the optimum compensation ratios (k_1^*, k_2^*) are found and tabulated (see Table IV.3) for various step input magnitudes (R).

A graph showing the relationship between the step input magnitudes and the compensation ratios (R vs. (k_1 and k_2)) is plotted (see Fig. IV.16). This curve can be divided into three regions:

$$\begin{aligned} &(1)\quad 0.1 \le R \le 0.6 \\ &(2)\quad 0.7 \le R \le 3.5 \\ &(3)\quad 3.5 < R \le 5. \end{aligned} \tag{IV.39}$$

where R is the magnitude of the step input. The flat second region indicates that if a suitable value for the compensation ratios is chosen, the system has satisfactory output responses for the input magnitudes in the second region. For example,

$$\begin{aligned} k_1 &= k_2 = 1.18 \ , \\ p_1 &= p_2 = 10.2764 \end{aligned} \tag{IV.40}$$

may be chosen as a (fixed) set of design parameters for the input magnitudes in Region 2. If this range covers most of the operational input values of the system and some degradation from the desired response can be tolerated[1] for the

---

[1] Since the system has a stable limit cycle, growing oscillations do not occur for any input magnitude.

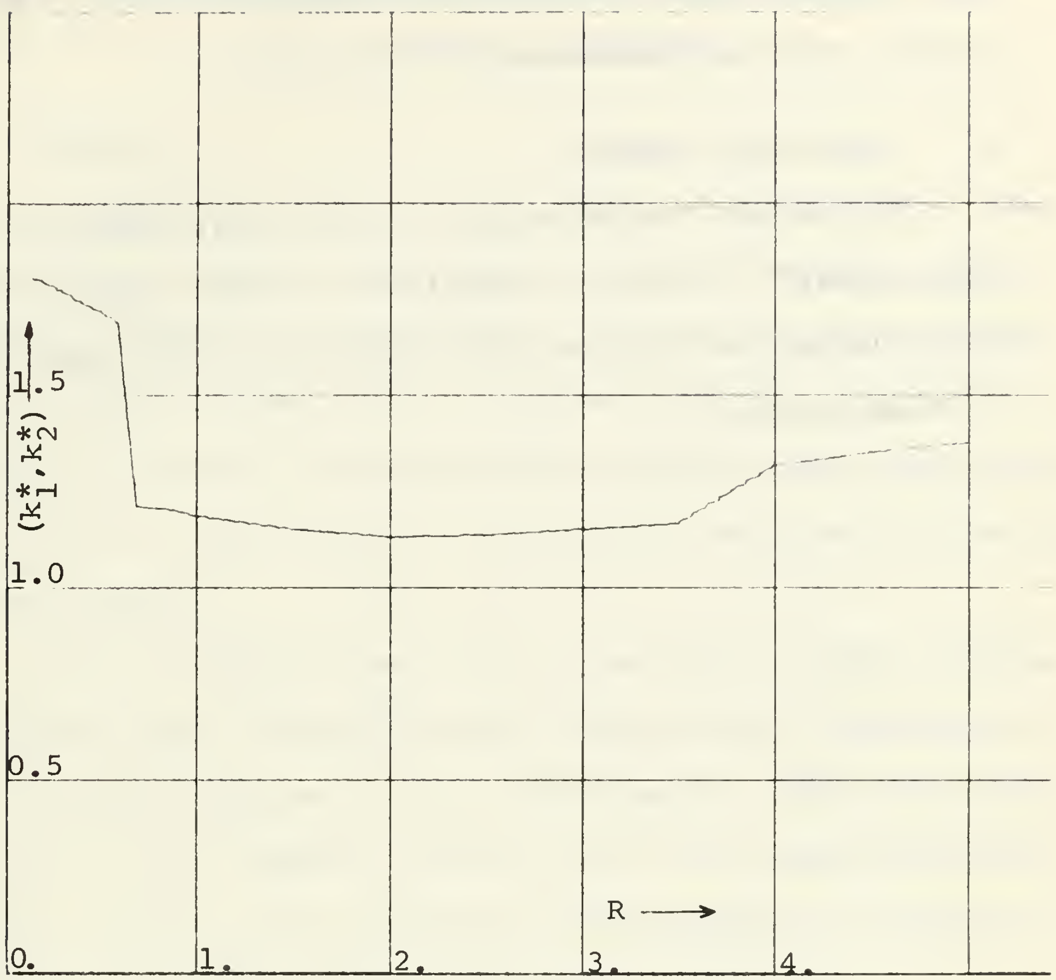


Fig. IV.16. EXAMPLE 11. Input magnitude (R) v.s. Optimum compensation ratios (K_1^* and K_2^*) for pole locations are fixed at $P_1 = p_2 = 10.264$.

other input magnitudes, this design is acceptable; otherwise another compensation scheme (probably adaptive or nonlinear) must be considered.

E. CONCLUDING REMARKS

The method "Optimization for the Best Response in the Time-Domain" is directly applicable to the design of sampled data systems and systems with one or more nonlinear elements or time delays.

V. A TIME-DOMAIN STABILITY ANALYSIS METHOD USING NUMERICAL TECHNIQUES

A. GENERAL

A numerical method for the stability analysis of feedback control systems is introduced in this chapter. This numerical method is applicable to a large variety of feedback control systems, but it may be valuable especially for the nonlinear and sampled-data systems and systems with time delay, since general and practical methods are rare in these fields.

If the stability analysis is carried out for a system which has been designed by using the method "Optimization for the Best Response in the Time-Domain", then only minor changes in the computer program yield the stability limit of the system; for this reason, this method is especially convenient for these types of design problems.

B. GENERAL PHILOSOPHY

The theory behind the numerical method is quite simple. It again depends on the idea of "shaping the transient response". The state equations of the system are integrated with respect to time by using a numerical method, the output of the system is observed during the integration (from zero time to final time - t_f), and the maximum and minimum points of the output are found. A performance index, to measure the sum of the differences of all the maxima and the differences of all the minima is defined

$$J(K) = \sum_{j=2}^{n} (x_{max}^{j} - x_{max}^{j-1})^2 + (x_{min}^{j} - x_{min}^{j-1})^2 , \qquad (V.1)$$

where K is assumed to be the only free variable, the superscript j is the counter for the maxima and minima, and n is the number of maxima and minima found in the time interval of interest.

Minimization of this performance index with respect to the free variable (K) yields the value of K^*, which puts the system at its stability limit, because with the K^* value, the output of the system becomes constant-magnitude oscillations.

For systems with two variables, a stability curve can be obtained by holding one of the variables at a constant value and applying the above procedure; at the end of the minimization, one point is found on the stability curve (in the system's parameter plane). By assigning other values to the fixed variable and minimizing the performance index with respect to the free variable, many points of the stability curve can be obtained and plotted.

The same idea can be extended to systems with more than two variables and stability surfaces or stability hypersurfaces can be obtained, but this may require too much computer time, and the practical use of stability hypersurfaces in n-dimensional Euclidean spaces is questionable. For practical design and analysis problems, the intersection of the stability hypersurface with a two-dimensional plane may yield the desired insight, and it can easily be obtained.

The terms in the performance index (values of the successive maxima and minima) are found by adding simple logic to the subroutine which calculates the value of the performance index. The maximum and minimum points of the response can be observed by different methods -- the simplest logic is discussed here. At the end of each integration step (of the system's state equations) the slope of the system's output is checked. If its sign is altered during the integration step, or if it takes the value zero, a maximum or a minimum has occurred. For a continuous output response every maximum must be followed by a minimum and vice versa; for this reason the performance index defined in Eq. (IV.1) may be altered to the simpler form

$$J(K) = \sum_{j=3}^{n}(x_m^j - x_m^{j-2})^2 , \qquad \text{(V.2)}$$

where x_m represents both the maxima and the minima, and n is the sum of the number of maximum and minimum points observed in the time interval of interest.

To demonstrate the application of the numerical stability analysis method three examples are given.

C. EXAMPLE 12

The stability limit of the chemical process with inherent time delay is to be found. The system (Fig.IV.1) and its state equations (IV.2) are given in Example 9 and will not be repeated here.

A performance index as defined in Eq. (V.2) is minimized with respect to the only free system parameter (K_c - the amplifier gain) and $K_c^* = 2.125$ is found for the stability limit. To check this result, the system is simulated with this gain value and the response shown in Fig. V.1 is obtained.

D. EXAMPLE 13

A sampled-data system (with a zero-order hold) was optimally compensated for a desired time response by using two free system parameters -- T (the sampling period) and K (the system gain) (see Example 10). The system is shown in Fig. IV.7(a).

The sampling period can be varied from 0.1 second to 1.5 seconds and the gain can be varied from zero to 15.

The value of the sampling period is varied in steps; it is first set to 0.1 then increased by 0.1 steps (up to 1.5). At each value of the sampling period, a performance index similar to the one defined in Eq. (V.2) is minimized

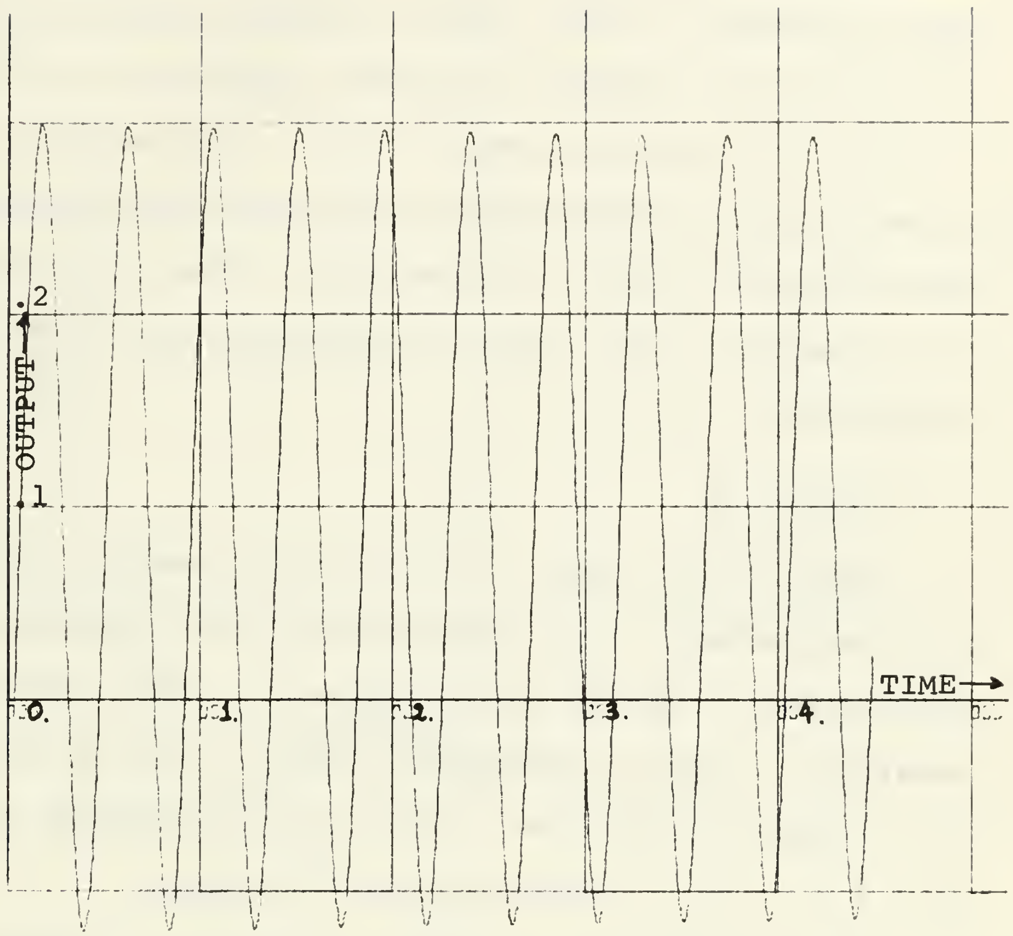


Fig. V.1. EXAMPLE 12. Digital simulation of the system at its stability limit (K_c = 2.125).

with respect to K and the stability curve in the T - K plane is plotted (see Fig. V.2). This curve simply gives the maximum allowable gain for a selected value of the sampling period, or vice versa. Several points on the stability curve are checked by digital simulation; for example, $T = 0.2$, $K = 10.34$ are read from Fig. V.2 and with these parameter values the system is simulated, and the time response shown in Fig. V.3 is obtained. Similarly, for $T = 1.5$ and $K = 1.74$, the time response shown in Fig. V.4 is obtained.

E. EXAMPLE 14

The stability limit of the system, shown in Fig.V.5(a) is to be studied. The uncompensated linear system is unstable when $K > 193.23$ (see Example 2). The system is compensated by using a tachometer, and $K_t = 0.4$ is selected for the desired response. The linear compensated system (with $K_t = 0.4$) is unstable when $K > 1020$.

The tachometer used in the system has a nonlinear characteristic and it saturates when its output reaches the saturation level (Z_m). In other words, the output of the tachometer is a function of the magnitude of its input and can be represented as (see Fig. V.5(b))

$$z(t) = \begin{cases} Z_m & \text{if} \quad K_t\dot{c}_s(t) \geq Z_m \\ -Z_m & \text{if} \quad K_t\dot{c}_s(t) \leq -Z_m \\ K_t\dot{c}_s(t) & \text{otherwise} \end{cases} \tag{V.3}$$

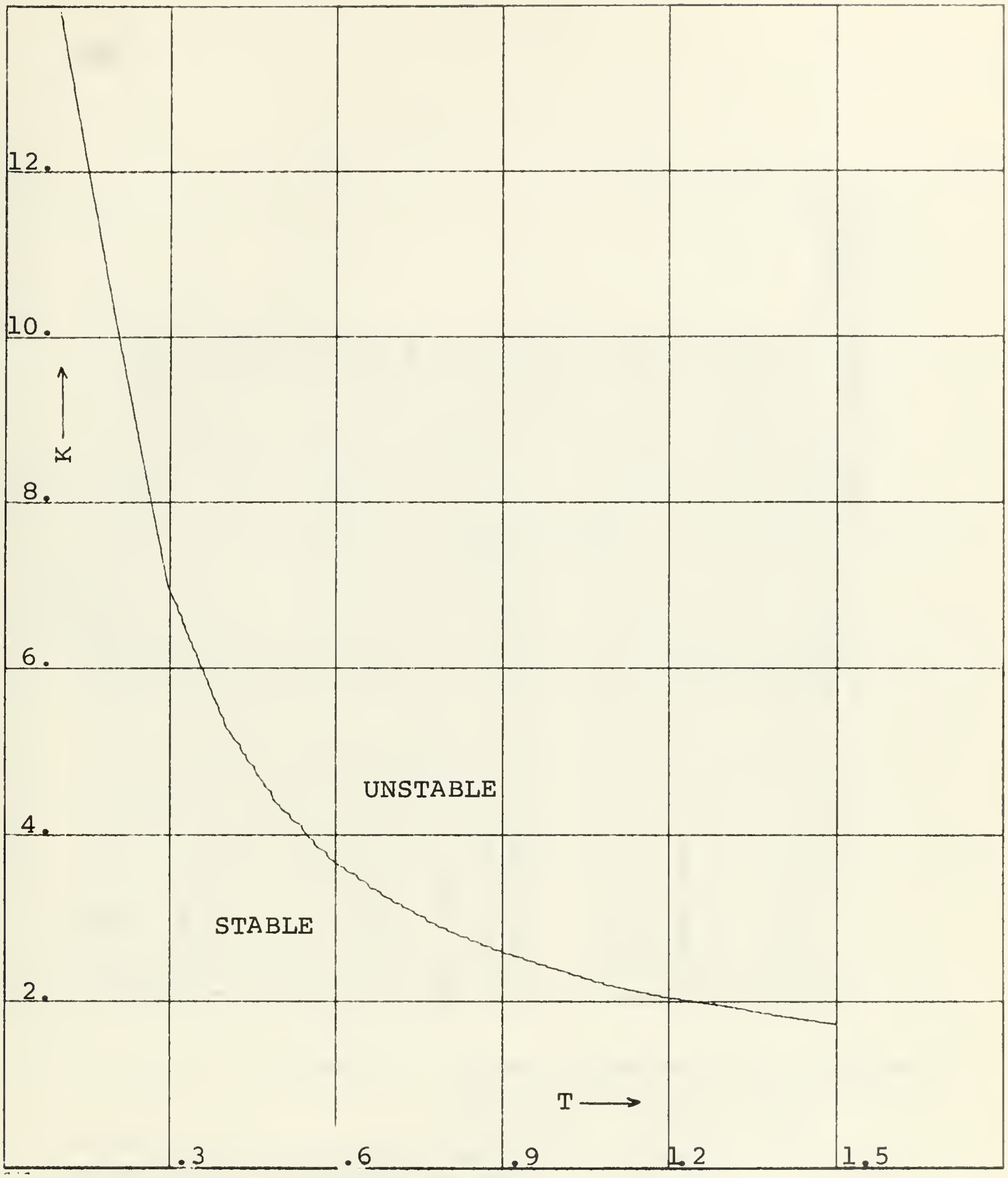


Fig. V.2. EXAMPLE 13. The stability curve in the "T - K" plane.

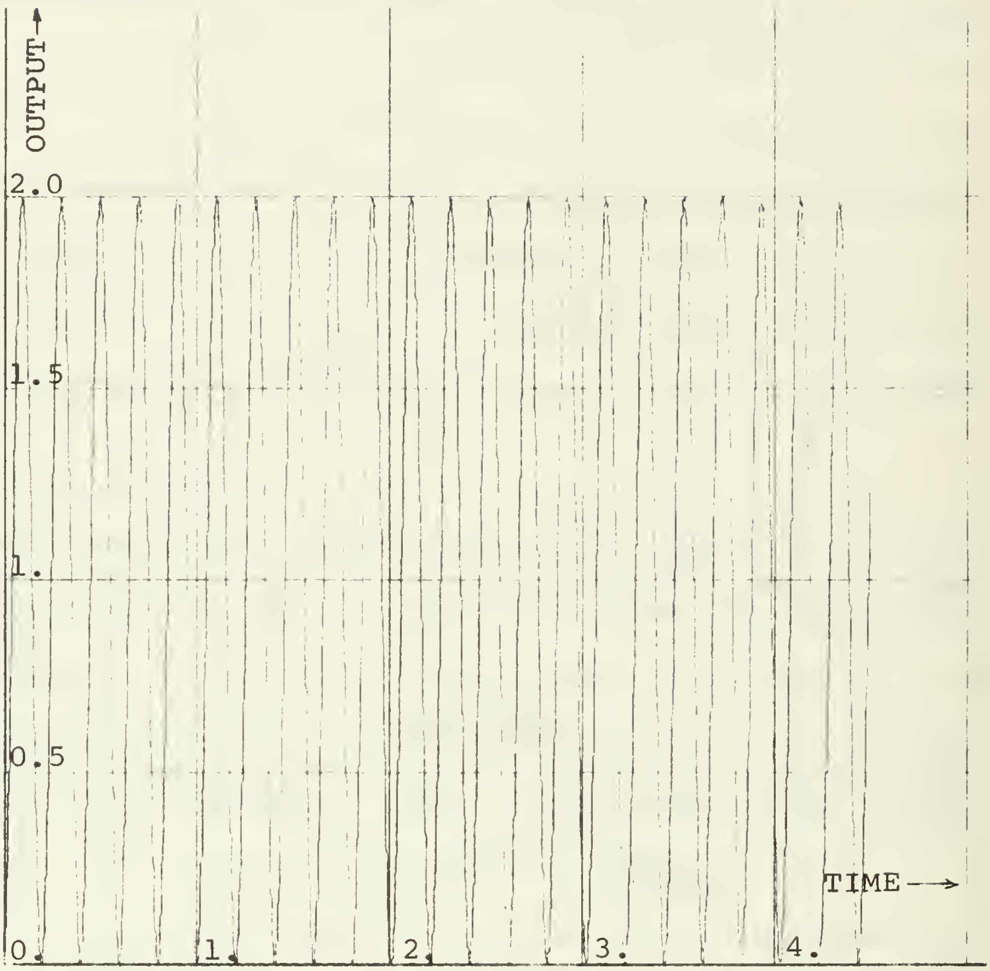


Fig. V.3. EXAMPLE 13. Time response of the system at one of its stability limits (T = 0.2, K = 10.34).

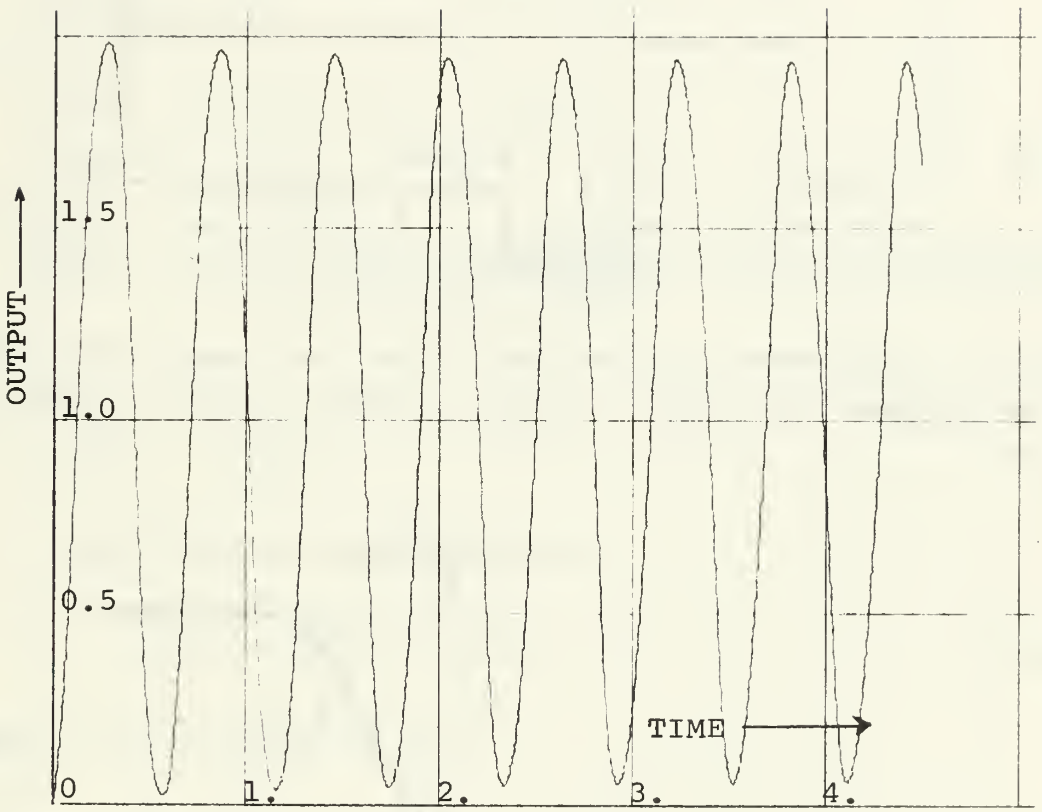
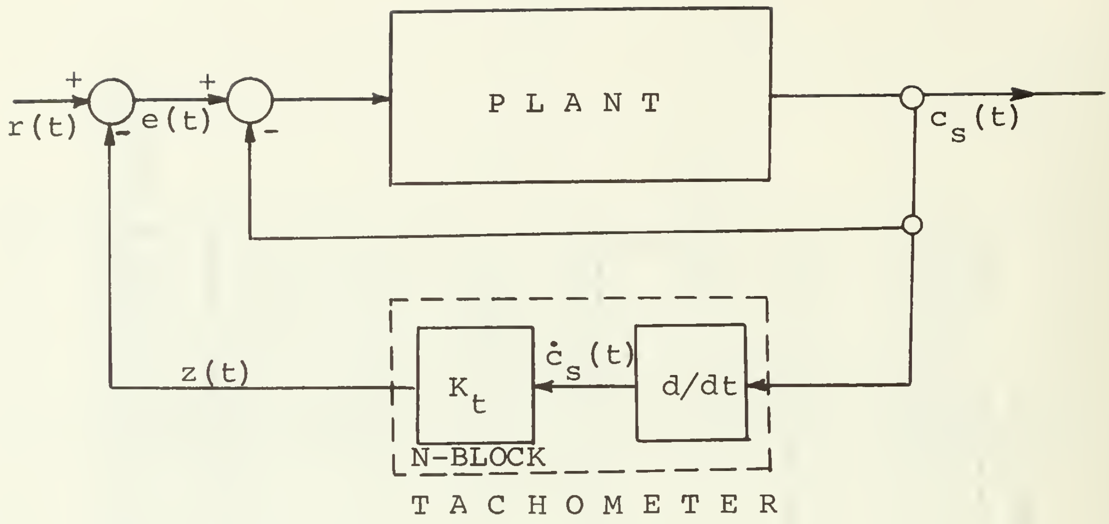
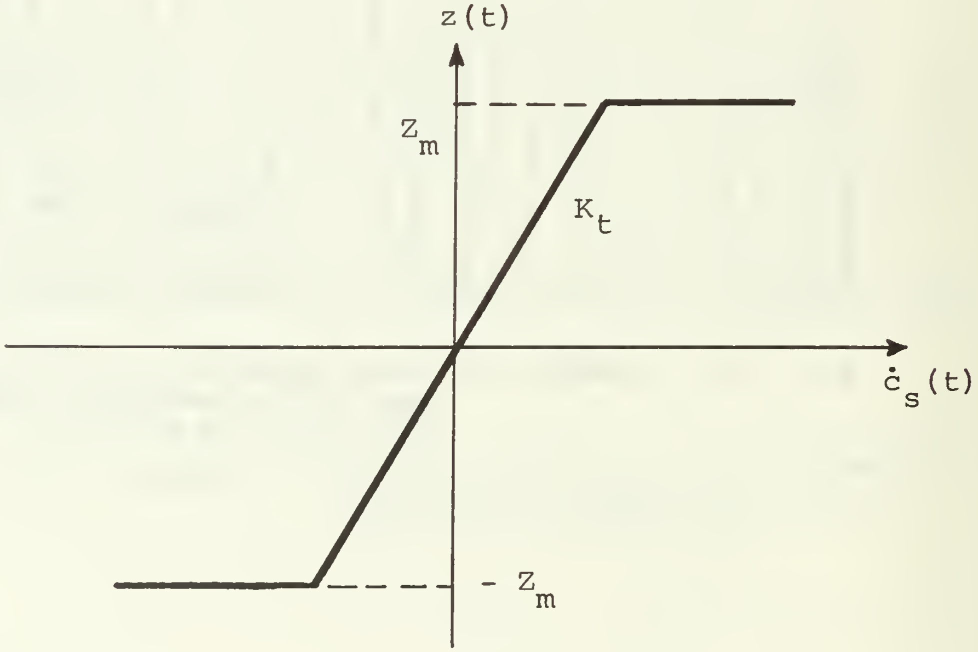


Fig. V.4. EXAMPLE 13. Time response of the system at one of its stability limits (T = 1.5 , K = 1.74).



(a) System (the plant is same as shown in Fig. II.8(a))



(b) N - Block

Fig.V.5. EXAMPLE 14.

For a given saturation level (Z_m), the stability of the system depends on the value of the gain (K) and also on the magnitude of the test (input) signal (R).

The state equations of the system, written directly from Fig. V.5(a), are

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= K(r(t) - z(t) - x_1(t)) - (50x_2(t) + 65x_3(t) + 16x_4(t)),
\end{aligned} \tag{V.4}$$

where the input is a step function with magnitude R, i.e.

$$r(t) = R \times \mathbb{1}(t) \ . \tag{V.5}$$

The output of the system is

$$c_s(t) = x_1(t) \ , \tag{V.6}$$

and its time derivative is

$$\dot{c}_s(t) = x_2(t) \ . \tag{V.7}$$

For a given step input magnitude (R), the state equations of the system are integrated by using a numerical method; after each integration step, the logic given in (V.3) is used to calculate z(t) ($Z_m = 0.5$ is used). The sign of $x_2(t)$ is also checked against its previous sign and the maximum and minimum points of the output ($x_1(t)$) are found. The performance index (Eq. (V.2)) is minimized with respect to (K). The result of this minimization is the maximum allowable gain for a given input magnitude (R) and for the given saturation level ($Z_m = 0.5$). After a minimization is completed, another magnitude for the step input is chosen and the minimization is repeated. The maximum gain values (K) for each magnitude of the step input (R) are stored and the stability curve in the R - K plane (see Fig. V.6(a)) is obtained (for $Z_m = 0.5$). When the input magnitude is small ($R \le 0.6$), the operation of the system is linear and the stability limit is at K = 1020. As the input magnitudes increase, the tachometer feedback becomes less effective, due to saturation. For example, when the step input magnitude (R) is 100, the system is unstable for K > 196.07; for this reason the stability curve (shown in Fig. V.6(a)) starts from K = 1020 for small input magnitudes, and asymptotically approaches the uncompensated system's stability limit (K = 193.23) as the input magnitudes increase. The same stability curve of Fig. V.6(a) is replotted to show the effect of magnitudes up to R = 100 (see Fig. V.6(b)).

As a different problem, the input magnitude is fixed at

$$R = 2.5 = \text{Constant} , \tag{V.8}$$

and the effect of the saturation level on the stability limit is studied.

The saturation level of the tachometer feedback (Z_m) is kept constant, and the minimization is carried out with

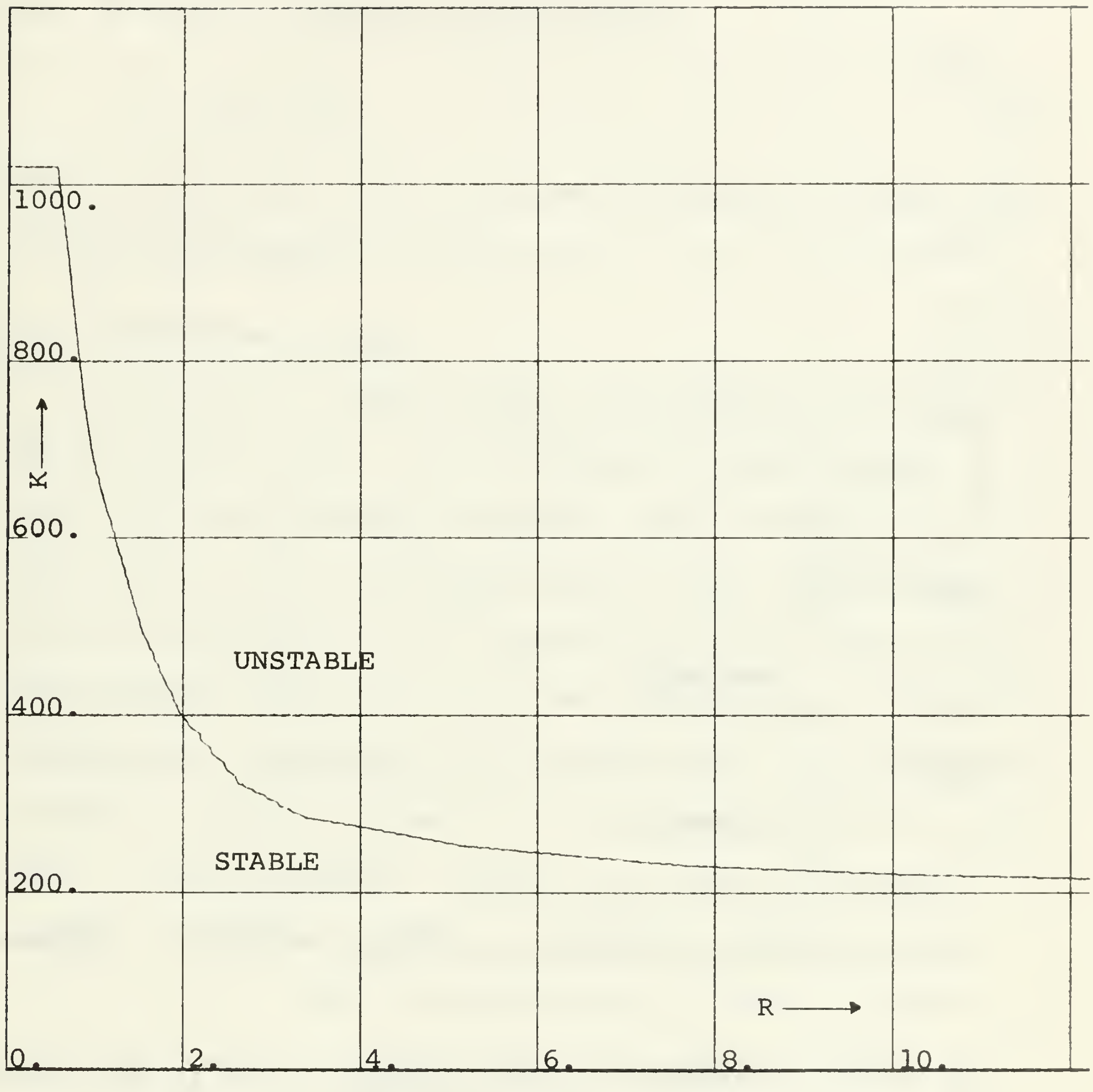


Fig. V.6(a). EXAMPLE 14. The stability curve in the "R - K" plane.

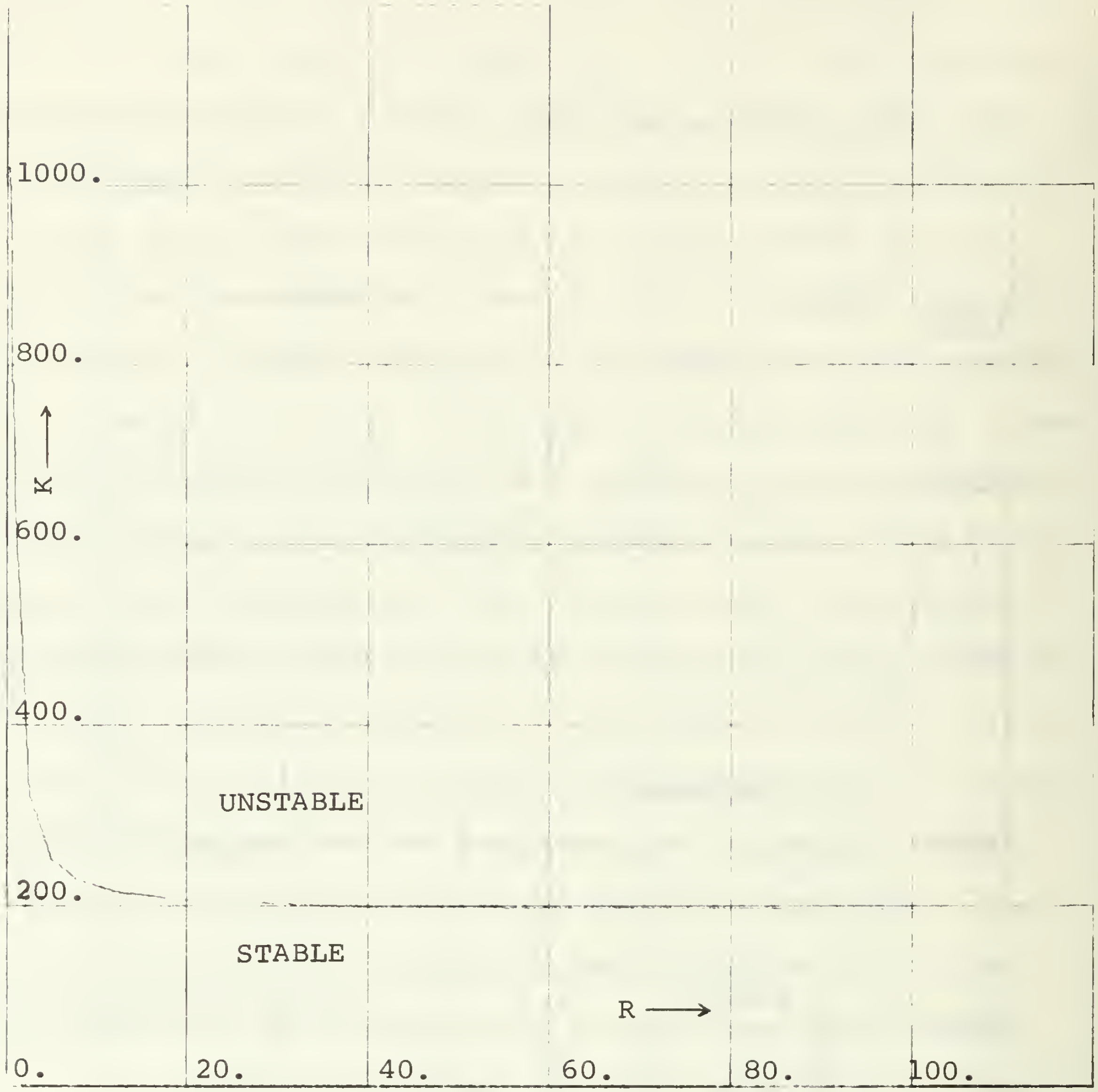


Fig. V.6(b). EXAMPLE 14. The stability curve in the extended "R - K" plane.

respect to K; when the value of K^* for a given Z_m is found, Z_m is changed to another value and the minimization is repeated. The maximum K values for each saturation level on the tachometer feedback, in the range

$$0.7 \le Z_m \le 5.7 \qquad (V.9)$$

are found and the stability curve in the Z_m - K plane (for R = 2.5) is plotted in Fig. V.7.

F. CONCLUDING REMARKS

Two words of caution about the computer programming are appropriate here. The performance index defined in Eq. (V.2) has at least two minima; one of these occurs when the system is overdamped and the time response has no critical points. In this situation both the value of the performance index and its gradient become zero, and this erroneously indicates a point on the stability curve. Some provision should be added to the computer program to avoid this erroneous result. Counting the number of maximum points in the time response, and finding new starting points if this number decreases below a certain predetermined constant, is one way of avoiding this difficulty.

Secondly, the performance index defined in Eq. (V.2) measures the differences in successive peaks and successive low points of the time response; but it is also a function of the number of maxima and minima which occur in the time interval of interest. This time interval may contain

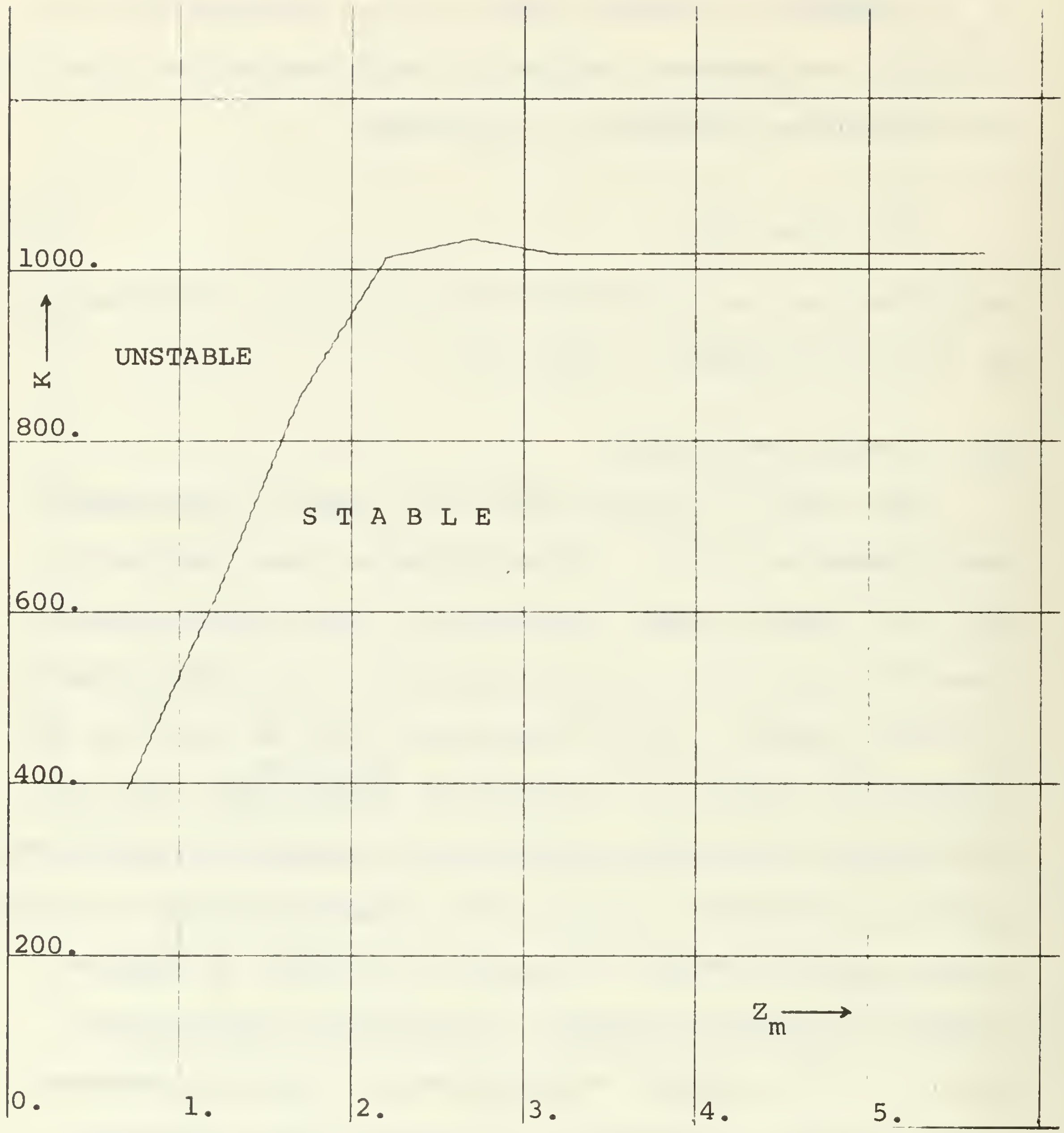


Fig. V.7. EXAMPLE 14. The stability curve in the "Z_m - K" plane.

different numbers of maxima and minima for different parameter values. To make the performance index independent of the number of maxima and minima, only a certain number of them may be included in the performance index, for example the third, fourth and fifth peaks and the low points of the time response may be measured and included in the performance index. Averaging the performance index over the total number of maxima and minima may also overcome this difficulty.

The two programming difficulties discussed above are not really serious and with a little care they can be overcome. The numerical method used to investigate the stability of the feedback control systems can be very attractive for the stability analysis of nonlinear and sampled-data systems and systems with time delay.

This method is an extension of the design method "Optimization for the Best Response in the Time-Domain". If a system is designed by using this method, the stability analysis can be carried out with little additional effort, since only minor changes (primarily in the performance index) convert the subroutine which was used in optimal design to the one to be used in the stability analysis.

VI. CONCLUSIONS

Two synthesis methods for the design of multi-parameter dynamic systems have been presented. The powerful mathematical tools of Optimal Control Theory have been applied to a variety of system design problems.

The first method, "Optimization for the Best Root Locations in the s-Domain" has been applied to linear system design problems. The second method, "Optimization for the Best Response in the Time-Domain" has been applied to the design of linear, nonlinear and sampled data systems and to a system with time delay. Eleven design examples have been considered; successful solutions of these examples may allow one to conclude, "These design methods provide fast, direct and efficient solutions to practical system design problems, and they can compete with any existing design methods."

The second method has been extended to a numerical stability analysis method, and its effectiveness has been demonstrated by three examples.

The methods presented in this thesis provide computerized design procedures for a wide class of dynamic systems, and hopefully, they provide one of the necessary bridges to narrow the existing gap between the theory and practice of system control.

```
C       C O M P U T E R   P R O G R A M  - - I
C
C     SAMPLE PROGRAM FOR LINEAR SYSTEM DESIGN,BY THE METHOD -
C        OPTIMIZATION FOR THE BEST ROOT LOCATIONS IN THE S-DOMAIN
C
        E X A M P L E  -  4
C
        DIMENSION XS(3),BU(3),BL(3),XMN(6)
        COMMON C,W,SIGD,WD,SIGLT2
C     V A R I A B L E S (Z-VECTOR)
C     # 1  SIGMA
C     # 2  P  (POLE LOCATION OF THE COMPENSATOR)
C     # 3  B (THE REMOTE ROOT)
C     IMPLICIT VARIABLES (LHS OF THE CONSTRAINING EQUATIONS (II-51))
C     C  AND  W ARE AS USED IN THE WRITE UP
C     # 4  A (THE CLOSE ROOT)
C     # 5  K (GAIN)
C     # 6  MINUSCULE K (THE COMPENSATION RATIO-- IN THE PRINT OUT'ALPHA')
C
C     ALPHA-VECTOR (#2 , #5 , #6 )
C     BETA-VECTOR ( #3 , #4 )
C
C     UPPER LIMITS
        BU(1)=2.
        BU(2)=0.15
        BU(3)=30.
C     LOWER LIMITS
        BL(1)=0.1
        BL(2)=0.0001
        BL(3)=8.
C     STARTING POINT
        XS(1)=1.162291
        XS(2)=0.08216614
        XS(3)=8.755328
C     NORM FOR THE STOPPING CRITERION
        EPS=0.01
C     DESIRED ROOT LOCATIONS
        SIGD=1.
        WD=2.
        SIGLT2=1.5
C
C     SUBROUTINE 'BOXPLX' (N.P.G.S.C.F. PROGRAM LIBRARY) FOR MINIMIZATION
C        OF FUNCTIONS WITH LINEAR AND NONLINEAR CONSTRAINTS
C     THE IMPORTANT INPUTS ARE : XS,BU,BL (AS DEFINED ABOVE)
C
    100 CALL BOXPLX(3,3,100,0,0.,XS,BU,BL,XMN,YMN,NMN,MMN)
C
C     THE IMPORTANT OUTPUTS ARE :
```

```
C       XMN : VALUES OF THE VARIABLES AT A MINIMUM
C       YMN : VALUE OF THE PERFORMANCE INDEX AT A MINIMUM
C
        WRITE(6,1) NMN,MMN
      1 FORMAT(2I6)
        IF(NMN.LE.-1) GO TO 130
        R=SQRT(YMN)
        WRITE(6,2) R
      2 FORMAT(//48H$$MINIMUM DISTANCE FROM THE DESIRED LOCATION =
     *F10.3,2H$$)
        WRITE(6,3)
      3 FORMAT(//T8,'D E S I R E D  L O C A T I O N ')
        WRITE(6,4) SIGD,WD
      4 FORMAT(//10H**SIGMA = F10.3,10H**OMEGA = F10.3,2H**)
        WRITE(6,5)
      5 FORMAT(//T8,'D E S I R E D  R O O T S')
        WRITE(6,4) XMN(1),W
        WRITE(6,6)
      6 FORMAT(//T8,'U N D E S I R E D  R O O T S')
        WRITE(6,7) XMN(3),XMN(4)
      7 FORMAT(//16H**REMOTE ROOT = F10.3,15H**CLOSE ROOT =
     *F10.5,2H**)
        WRITE(6,8)
      8 FORMAT(//T8,'OPTIMUM DESIGN PARAMETERS')
        WRITE(6,9) XMN(5),XMN(2),XMN(6)
      9 FORMAT(//9H**GAIN = F10.3,6H**P = F10.3,10H**ALPHA =
     *F10.3,2H**)
        WRITE(6,10)
     10 FORMAT(//40H**************************************** )
C     CHECK IF THE DOMINANT ROOTS ARE IN A CLOSE NEIGHBORHOOD OF THE-
C       DESIRED ROOT LOCATIONS
        IF(R.LT.EPS) GO TO 14
C     CHECK,AND IF NECESSARY RELAX SIGLT-1 (NOTE : SIGLT-1 = BU(3))
        IF(ABS(XMN(3)-BL(3)).GT.EPS) GO TO 11
        BL(3)=BL(3)-1.
        IF(BL(3).LE.1.) GO TO 11
        GO TO 12
C     CHECK,AND IF NECESSARY RELAX SIGLT-2
     11 IF(ABS(XMN(4)-SIGLT2).GT.EPS) GO TO 14
    130 SIGLT2=SIGLT2+0.1
        IF(SIGLT2.GE.2.) GO TO 14
C     AFTER RELAXATION,START THE NEW MINIMIZATION FROM THE LAST MINIMUM
     12 DO 13 I=1,3
     13 XS(I)=XMN(I)
        GO TO 100
C     ALL POSSIBLE RELAXATION IS COMPLETED,OR
C       THE DESIRED ROOTS ARE LOCATED AT THE DESIRED LOCATIONS
C     DIGITAL SIMULATION - BY USING OPTIMUM PARAMETERS
```

```
   14 CALL EPLT48(XMN)
      WRITE(6,15)
   15 FORMAT(//T8,'ERKAL TUZGIRAY')
      RETURN
      END



      FUNCTION KE(X)
      DIMENSION X(6)
      COMMON C,W,SIGD,WD,SIGLT2
C
C  IMPLICIT CONSTRAINTS (TYPE-C)  ARE CHECKED
C  THIS FUNCTION IS CALLED BY SUBROUTINE 'BOXPLX'
C  SEE EQUATIONS (II-51) AND (II-52)
C  IF ALL CONSTRAINTS ARE SATISFIED--SET KE=0;IF NOT SET KE=-1
      KE=0
      X(4)=12.+X(2)-X(3)-2.*X(1)
      IF(X(4).GT.SIGLT2) GO TO 1
      C=36.+12.*X(2)-2.*X(1)*(X(4)+X(3))-X(4)*X(3)
      WSQ=C-X(1)**2
      IF(WSQ.LT.0.) GO TO 1
      W=SQRT(WSQ)
      X(5)=X(4)*X(3)*C/X(2)
      IF(X(5).LT.425.) GO TO 1
      X(6)=X(5)/(2.*X(4)*X(3)*X(1)+C*(X(4)+X(3))-36.*X(2))
      IF(X(6).LT.1..OR.X(6).GT.10.) GO TO 1
      RETURN
    1 KE=-1
      RETURN
      END



      FUNCTION FE(X)
      DIMENSION X(3)
      COMMON C,W,SIGD,WD,SIGLT2
C
C  PERFORMANCE INDEX (J=FE) -- EQUATION (II-53) -- IS CALCULATED
C  THIS FUNCTION IS CALLED BY SUBROUTINE 'BOXPLX'
      ALEG=12.+X(2)-X(3)-2.*X(1)
      C=36.+12.*X(2)-2.*X(1)*(ALEG+X(3))-ALEG*X(3)
      W=SQRT(C-X(1)**2)
      FE=(SIGD-X(1))**2+(WD-W)**2
      RETURN
      END
```

```
      SUBRCUTINE EFLT48(A)
      DIMENSION A(6),XX(600),YY(600),ZZ(600)
      REAL*8 X(6),XDOT(6),ITITLE(12),T,DT
      REAL*4 LAB/4H    /
C
C   DIGITAL SIMULATICN WITH CPTIMUM PARAMETER VALUES
C      AFTER ALL RELAXATIONS ARE CCMPLETED
C
C   INITIAL CCNDITICNS
      DC 1 I=1,6
    1 X(I)=1.D-12
C   INITIAL TIME FCR THE INTEGRATICN
      T=0.
C   INTEGRATICN STEP SIZE
      DT=0.01
      NT=0
      H3=A(5)/A(6)
      B0=A(5)*A(2)
      B1=36.*A(2)+H3
      B2=36.+12.*A(2)
      B3=12.+A(2)
      H4=B0-H3*B3
      DO 7 I=1,600
C   THE STATE EQUATICNS OF THE COMPENSATED SYSTEM
    2 XDOT(1)=X(2)
      XDCT(2)=X(3)
      XDOT(3)=X(4)+H3
      XDOT(4)=H4-(B0*X(1)+B1*X(2)+B2*X(3)+B3*X(4))
C   THE STATE EQUATICNS OF THE SECCND-CRDER MODEL
      XDOT(5)=X(6)
      XDOT(6)=5.*(1.-X(5))-2.*X(6)
C   SUBROUTINE 'RKLDEQ' (N.P.G.S.C.F. PROG. LIBRARY) FOR INTEGRATION
      S=RKLDEQ(6,X,XDOT,T,DT,NT)
      IF(S-1.) 5,2,6
    5 WRITE(6,4)
    4 FORMAT(//T8,'INTEGRATICN TRCUBLE IN PLT48')
      RETURN
    6 TT=I
      XX(I)=TT*DT
      YY(I)=X(1)
      ZZ(I)=X(5)
    7 CONTINUE
      READ(5,3) (ITITLE(I),I=1,12)
    3 FORMAT(6A8)
```

```
C   SUBROUTINE ' DRAW ' (N.P.G.S.C.F. PROG. LIBRARY) FOR PLOTS
        CALL DRAW(600,XX,YY,1,0,LAB,ITITLE,0,0,0,0,0,0,8,8,1,L)
        WRITE(6,8) L
        CALL DRAW(600,XX,ZZ,3,0,LAB,ITITLE,0,0,0,0,0,0,8,8,1,L)
        WRITE(6,8) L
      8 FORMAT(I6)
        RETURN
        END
```

```
C       C O M P U T E R   P R O G R A M   -   II
C
C    SAMPLE PROGRAM FOR SYSTEM DESIGN,BY THE METHOD -
C       OPTIMIZATION FOR THE BEST RESPONSE IN THE TIME DOMAIN
C
C       E X A M P L E  -  10
C
        IMPLICIT REAL*8(A-H,P-Z)
        REAL*4 CC
        DIMENSION A(2),AST(2),ALL(2),AUL(2),AB(2),B(2),G(2),FF(3)
C
C    DESIGN OF A SAMPLED DATA SYSTEM WITH A ZERO ORDER HOLD
C    V A R I A B L E S (ALPHA - VECTOR)
C    # 1  GAIN
C    # 2 SAMPLING PERIOD
C    UPPER LIMITS ON THE VARIABLES
        AUL(1)=15.
        AUL(2)=1.5
C    LOWER LIMITS ON THE VARIABLES
        ALL(1)=0.1
        ALL(2)=0.1
C    STARTING POINT
        AST(1)=0.5
        AST(2)=1.
        EPS=1.D- 4
        CALL ERGRAD(2,1,-1,AUL,ALL,AST,EPS,F,AB)
        CC=AB(1)
        NM=AB(2)/0.051
        CALL EPLT10(NM,CC)
        WRITE(6,1)
      1 FORMAT(//T8,'ERKAL TUZGIRAY')



        SUBROUTINE ERGRAD(NPAR,MSTOP,JMAX,AUL,ALL,AST,EPS,F,AB)
        IMPLICIT REAL*8(A-H,P-Z)
        DIMENSION A(8),AST(8),ALL(8),AUL(8),AB(8),B(8),G(8),FF(9),
       $BSTA(8),BETA(8),DP(8)
C
C        D I S C R E T E   G R A D I E N T   M I N I M I Z A T I O N
C
C    TO FIND MINIMA OF A SCALAR FUNCTION OF MANY VARIABLES
C
C    CALLING STATEMENT OF THE MAIN PROGRAM CONTAINS 9 VARIABLES
C    THE FIRST THREE VARIABLES ARE INTEGERS
C       I N P U T S
C    NPAR : #OF FREE PARAMETERS,(1<NPAR<9)
```

```
C  MSTOP: #OF LOCAL MINIMA TO BE FOUND STARTING FROM DIFFERENT-
C     STARTING POINTS,(THE NEW STARTING POINT IS GENERATED IN-
C     ERGRAD BY CHANGING THE VALUES OF #1 AND #2 PARAMETERS)
C  JMAX : #OF TIMES THE GRAD STEP SIZE DIVIDED BY TWO
C     IF (JMAX<0 IS USED) JMAX=10 WILL BE USED IN THE PROGRAM
C  AUL  : UPPER LIMITS ON THE PARAMETERS (NPAR-DIM VECTOR)
C  ALL  : LOWER LIMITS ON THE PARAMETERS (NPAR-DIM VECTOR)
C  AST  : STARTING POINT                 (NPAR-DIM VECTOR)
C  EPS  : ACCURACY CONSTANT (TO CHECK THE VALUE OF GRAD)
C     O U T P U T S
C  EACH STEP IN THE NEGATIVE GRAD DIRECTION WILL BE PRINTED
C  FIRST LINE CONTAINS 2 NUMBERS WHICH ARE THE VALUE OF THE-
C     FUNCTION AND THE ABS MAGNITUDE OF THE GRAD VECTOR
C  SECOND LINE CONTAINS NPAR PARAMETER VALUES AT THAT STEP.
C  THIRD LINE CONTAINS THE INDIVIDUAL GRAD COMPONENTS-
C     CORRESPONDING TO THE PARAMETER VALUES WRITTEN -
C     IN THE PREVIOUS LINE
C  OPTIMUM PARAMETER VALUES WILL BE PPRINTED IN THE SAME -
C     FORM,WHEN A MINIMUM IS FOUND
C  VALUE OF THE FUNCTION (F) AND OPTIMUM PARAMETER VALUES (AB)-
C     AT A LOCAL MIN (IF MSTOP>1--AT THE MSTOP'TH MIN) -
C     WILL BE RETURNED TO THE CALLING PROGRAM
C  THE USER MUST SUPPLY A SUBROUTINE 'ERKAL(A,F,NTR)' TO CAL-
C     CULATE THE VALUE OF THE FUNCTION TO BE MINIMIZED
C  WHERE:'A' IS THE CURRENT VALUES OF THE NPAR-PARAMETERS-
C     AND 'F' IS THE CORRESPONDING VALUE OF THE FUNCTION-
C     'NTR' MUST BE SET  TO A POSITIVE INTEGER AT THE BEGINNING-
C     OF THE SUBROUTINE,THEN IF A TROUBLE IN THE FUNCTION -
C      EVALUATION OCCURS, IT MUST BE SET TO A NEGATIVE INTEGER -
C     IN THIS CASE MINIMIZATION PROCESS WILL BE RESTARTED -
C     FROM A NEW STARTING POINT--SUPPLIED BY ERGRAD
C  ONLY THE VALUES OF #1 AND #2 PARAMETERS WILL BE CHANGED TO -
C     FIND THE NEW STARTING POINT,FOR THIS REASON THE MOST-
C     EFFECTIVE TWO VARIABLES SHOULD BE USED AS #1 & #2
C  NORMAL EXIT FOR A LOCAL MINIMUM OCCURS IF :
C  1 - THE GRAD STEP SIZE DIVIDED BY 2,MORE THAN JMAX TIMES,
C  2 - ABS GRAD MAGNITUDE IS LESS THAN EPS
C
C  ALL INPUT AND OUTPUT VARIABLES (ACCEPT THE INTEGERS)-
C     MUST BE DOUBLE PRECISSION
C
      IF(JMAX.LT.0) JMAX=10
  999 NSTOP=1
      NRUN=0
      L=4
      J=1
      MPAR=NPAR+1
      C=0
```

```
C  FINDING THE GRAD STEP SIZES FOR EACH VARIABLE
      DO 20 I=1,NPAR
      DDD=AUL(I)-ALL(I)
      BSTA(I)=DDD/10.
   20 DP(I)=DDD/1.D 4
  200 L=L+1
      JA=0
      DO 1 I=1,NPAR
      A(I)=AST(I)
    1 BETA(I)=BSTA(I)
C  CALCULATE THE VALUE OF 'F' AND GRAD VECTOR
  100 DO 102 I=1,NPAR
      IF(I.EQ.1) GO TO 101
      B(I-1)=A(I-1)
      A(I-1)=A(I-1)+DP(I-1)
  101 CALL ERKAL(A,F,NTR)
      IF(NTR.LT.0) GO TO 129
      FF(I)=F
      IF(I.EQ.1) GO TO 110
      A(I-1)=B(I-1)
      GO TO 102
  110 IF(NRUN.EQ.0) GO TO 102
      IF(FF(1).GE.FB) GO TO 500
  102 CONTINUE
C  SUCSESFUL STEP TOWARD MINIMUM
    2 FB=FF(1)
      NRUN=1
C  CALCULATE ABS MAG OF THE GRAD CHECK AGAINST 'EPS'
      GSQ=0.
      DO 103 I=1,NPAR
      G(I)=(FF(I+1)-FF(1))/DP(I)
  103 GSQ=GSQ+G(I)**2
      GABS=DSQRT(GSQ)
      DO 3 I=1,NPAR
    3 AB(I)=A(I)
      IF(GABS.LT.EPS) GO TO 6
      GO TO 4
C  UNSUCSESFUL STEP,START DIVIDING THE STEP SIZES ONE AT A TIME
  500 JA=JA+1
      IF(JA.LE.NPAR) GO TO 501
      JA=1
      J=J+1
      IF (J.GT.JMAX) GO TO 10
  501 BETA(JA)=BETA(JA)/2.
      WRITE(6,502) JA,J
  502 FORMAT(/T4,6H*BETA( I2,19H ) IS DIVIDED BY 2- I2,6H TIMES ,/)
    4 WRITE(6,15) FF(1),GABS
   15 FORMAT(//T15,2D15.7)
```

```
      WRITE(6,14) (A(I),I=1,NPAR)
      WRITE(6,14) (G(I),I=1,NPAR)
   14 FORMAT(8D15.7,//)
      DO 5 I=1,NPAR
      A(I)=AB(I)-G(I)*BETA(I)
      IF(A(I).LT.ALL(I)) A(I)=ALL(I)
      IF(A(I).GT.AUL(I)) A(I)=AUL(I)
    5 CONTINUE
      GO TC 100
C  FINDING A NEW STARTING PCINT
  129 WRITE(6,17)
   17 FORMAT(//T8,'TROUBLE IN FUCTION EVALUATION',//)
  130 IF(L.EQ.1) GO TC 131
      IF(L.EQ.2) GC TO 132
      IF(L.EQ.3) GC TC 133
      XLL=(AUL(1)-ALL(1))/2.
      C=C+1.
      D=4.*C*BSTA(1)
      D1=4.*C*BSTA(2)
      IF(D.GT.XLL) GO TO 12
      AST(2)=AST(2)+D
      AST(1)=AST(1)+D1
      GO TC 134
  131 AST(2)=AST(2)-D
      AST(1)=AST(1)+D1
      GO TC 134
  132 AST(2)=AST(2)-D
      AST(1)=AST(1)-D1
      GO TC 134
  133 AST(2)=AST(2)+D
      AST(1)=AST(1)-D1
  134 NRUN=0
      DC 135 I=1,2
      IF(AST(I).LT.ALL(I)) AST(I)=ALL(I)
  135 IF(AST(I).GT.AUL(I)) AST(I)=AUL(I)
      J=0
      GO TC 200
    6 WRITE(6,7)
    7 FORMAT(//T20,'A LOCAL MINIMUM IS FOUND',//)
      WRITE(6,15) FF(1),GABS
      WRITE(6,8)
    8 FORMAT(//T20,'OPTIMUM PARAMETER VALUES AT A LOCAL MINIMUM',/)
      WRITE(6,14) (A(I),I=1,NPAR)
      WRITE(6,16)
   16 FORMAT(//T20,46H**********************************************,//)
      NSTOP=NSTOP+1
      IF(NSTOP.GT.MSTCP) GC TC 12
      GO TO 130
```

```
10 WRITE(6,11) J
11 FORMAT(//43H****GRADIENT IS NOT ZERO BUT BETA DIVIDER =  I5)
   GO TO 6
12 WRITE(6,13)
13 FORMAT(//T8,'SEARCH IS COMPLETED',/)
   RETURN
   END



   SUBROUTINE ERKAL(A,F,NTR)
   IMPLICIT REAL*8(A-H,P-Z)
   DIMENSION A(2),X(5),XDOT(5)
C
C  CALCULATES THE VALUE OF THE PERFORMANCE INDEX
C
C  STEP INPUT
   U=1.0
C  INITIAL CONDITIONS ARE ZERO
   DO 1 I=1,5
 1 X(I)=0.0
   NTR=10
C  INITIAL TIME FOR INTEGRATION
   T=0.0
C  INTEGRATION STEP SIZE
   DT=0.05
   NT=0
C  SAMPLING COUNTER
   N=0
C  DESIRED ROOT LOCATIONS FOR THE MODEL RESPONSE
   WND=0.725
   SIG=0.314
C  #1 VARIABLE IS THE SYSTEM GAIN
   C=A(1)
C  #2 VARIABLE IS THE SAMPLING PERIOD
   DTK=0.051
   SEMPP=A(2)/DTK
   NM=SEMPP
   E=1.
   DO 7 I=1,300
C  THE STATE EQUATIONS OF THE SYSTEM
 2 XDOT(1)=X(2)
   XDOT(2)=C*E-X(2)
C  THE MODEL RESPONSE
   XDOT(3)=X(4)
   XDOT(4)= -X(3)*WND**2-2.*SIG*X(4)+(WND**2)*U
C  THE PERFORMANCE INDEX
```

```
      XDOT(5)=(X(1)-X(3))**2
C   SUBROUTINE 'RKLDEQ' (LIBRARY SUBROUTINE) FOR INTEGRATION
      Q=RKLDEQ(5,X,XDOT,T,DT,NT)
      IF(Q-1.) 4,2,6
    4 NTR=-10
      RETURN
    6 N=N+1
      IF(N.NE.NM) GO TO 7
      N=0
      E=U-X(1)
    7 CONTINUE
      F=X(5)
      RETURN
      END



      SUBROUTINE EPLT10(NM,C)
      DIMENSION XX(400),YY(400),ZZ(400)
      REAL*8 X(4),XDOT(4),T,DT,ITITLE(12)
      INTEGER LAB/4H    /
C
C   DIGITAL SIMULATION FOR OPTIMAL DESIGN OF
C       A SAMPLED DATA SYSTEM WITH ZERO ORDER HOLD
C   A SECOND ORDER CONTINUOUS MODEL IS USED IN THE  --
C       OPTIMIZATION TO REPRESENT THE DESIRED RESPONSE
C   SYSTEM PARAMETERS ARE THE SAMPLING PERIOD AND THE GAIN
C
      T=0.
      DT=0.05
      NT=0
      N=0
      WND=0.725
      SIG=0.314
      E=1.
      DO 1 I=1,4
    1 X(I)=0.
      DO 7 I=1,400
    2 XDOT(1)=X(2)
      XDOT(2)=C*E-X(2)
      XDOT(3)=X(4)
      XDOT(4)=-X(3)*WND**2-2.*SIG*X(4)+WND**2
C   SUBROUTINE 'RKLDEQ' (LIBRARY SUBROUTINE) FOR INTEGRATION
      S=RKLDEQ(4,X,XDOT,T,DT,NT)
      IF(S-1.) 4,2,6
    4 WRITE(6,3)
    3 FORMAT(//T8,'INTEGRATION TROUBLE')
```

```
      RETURN
    6 CT=I
      XX(I)=CT*DT
      YY(I)=X(1)
      ZZ(I)=X(3)
      N=N+1
      IF(N.NE.NM) GC TC 7
      N=0
      E=1.-X(1)
    7 CONTINUE
      READ(5,5) (ITITLE(I),I=1,12)
    5 FORMAT(6A8)
C   SUBROUTINE ' DRAW ' (LIBRARY SUBROUTINE) FOR PLOTS
      CALL DRAW(400,XX,YY,1,0,LAB,ITITLE,0,0,0,0,0,0,8,8,1,L)
      WRITE(6,8) L
    8 FORMAT(I6)
      CALL DRAW(400,XX,ZZ,3,0,LAB,ITITLE,0,0,0,0,0,0,8,8,1,L)
      WRITE(6,8) L
      RETURN
      END
```

BIBLIOGRAPHY

1. Box, M. J., "A New Method of Constrained Optimization and a Comparison with Other Methods," The Computer Journal, April 1965.

2. Chestnut, H., "Bridging the Gap in Control - Status 1965," IEEE Transactions on Automatic Control, April 1965.

3. Coughanowr, D. R., and Koppel, L. B., Processes Systems Analysis and Control, McGraw-Hill Book Co., New York, 1965.

4. Hilleary, R. R., "SUBROUTINE BOXPLX," N.P.G. School Computer Facility Subroutine Library, January 1966.

5. Kirk, D. E., Optimal Control Theory: An Introduction, Prentice-Hall, Inc., Englewood Cliffs, New Jersey. (To be published in 1970).

6. Leondes, C. T., Computer Control System Technology, McGraw-Hill Book Co., New York, 1961.

7. Newton, G. C., Gould, L. A. and Kaiser, J. F., Analytical Design of Linear Feedback Controls, J. Wiley and Sons, Inc., New York, 1957.

8. Šiljak, D. D., Nonlinear Systems -- The Parameter Analysis and Design, J. Wiley and Sons, Inc., New York, 1969.

9. Thaler, G. J., and Brown, R. G., Analysis and Design of Feedback Control Systems, McGraw-Hill Book Co., Inc. New York, 1960.

10. NASA Contractor Report 617, Algebraic Methods for Dynamic Systems, by G. J. Thaler, D.D. Šiljak and R.C. Dorf, November 1966.

11. Thaler, G. J. and Pastel, M. P., Analysis and Design of Nonlinear Feedback Control Systems, McGraw-Hill Book Co., Inc., New York, 1962.

12. Truxal, J. G., Automatic Feedback Control System Synthesis, McGraw-Hill Book Co, Inc., New York, 1955.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Professor G. J. Thaler, Code 52Tr Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	10
4.	Professor D. E. Kirk, Code 52Ki Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	3
5.	LT Erkal Tüzgiray, TURKISH NAVY Hüseyin Rahmi Bey Sokak 6/1 Heybeliada, İstanbul TURKEY	4
6.	İstanbul Teknik Üniversitesi Elektrik Fakültesi Taşkişla, İstanbul TURKEY	1
7.	Orta-Dogu Teknik Üniversitesi Elektrik Fakültesi Ankara, TURKEY	1
8.	Karadeniz Teknik Üniversitesi Elektrik Fakültesi Trabzon, TURKEY	1
9.	Professor N. Tanyolaç Dean of Research School of Engineering Robert College Bebek, İstanbul TURKEY	1
10.	Deniz Kuvvetleri Komutanliği Personel Eğitim Şb. Müdürlüğü Anakara, TURKEY	1

No. Copies

11. Deniz Harb Okulu Komutanliği — 1
Heybeliada, İstanbul
TURKEY

12. Deniz Makine Sinif Okullari Komutanliği — 1
Derince, Kocaeli
TURKEY

13. Mr. H. Karl Bouvier — 1
Spacecraft Control Section
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

14. Dr. R. C. Dorf, Dean — 1
School of Engineering
Ohio University
Athens, Ohio 45701

15. Dr. P. Kokotovic — 1
Coordinated Science Laboratories
University of Illinois
Urbana, Illinois 43078

16 Dr. D. D. Šiljak — 1
Department of Electrical Engineering
University of Santa Clara
Santa Clara, California 93153

17. Mr. J. Karmakar — 1
Department of Electrical Engineering
University of Santa Clara
Santa Clara, California 93153

18. Commander, — 1
Naval Ordnance Systems Command Hqs.
Department of the Navy
Washington, D. C. 20360

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Optimal Design of Multi-Parameter Dynamic Systems

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*

Master's Thesis; October 1969

5. AUTHOR(S) *(First name, middle initial, last name)*

LT. Erkal Tüzgiray

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1969	138	12

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

Two design methods for multi-parameter dynamic systems are proposed. They are intended to eliminate the limitations and disadvantages of the existing design methods. The powerful mathematical tools of optimal control theory are applied to the practical design problems of classical control.

The first method is intended for linear systems only; the design problem is solved in the s-domain, by finding "the best root locations" of the system's characteristic equation. In the second method, the design problem is solved by finding "the best response" of the system in the time domain. The second method is applicable to a wide range of dynamic systems; it can be used to synthesize linear, nonlinear and sampled-data systems, and systems with time delay. This method is also extended to a numerical stability analysis procedure.

Fourteen examples are presented to illustrate the applications of the methods.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
System Design						
System Synthesis						
Control						
Automatic Control						
Feedback System						
Optimal Design						
Stability						
Time Delay						

DD FORM 1 NOV 65 1473 (BACK)
S/N 0101-807-6821

Unclassified
Security Classification

A-31409